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A SURVEY OF FAILURE THEORIES OF ISOTROPIC AND ANISOTROPIC MATERIALS

R. S. Sandhu

Air Force Flight Dynamics Laboratory
Wright-Patterson Air Force Base, Ohio

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R. S. SANDHU

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A survey of various theories of strength for isotropic and anisotropic materials is presented. For anisotropic materials theories of failure are broadly classified as theories with or without distinct failure modes. The former class includes the criteria based upon maximum stress and strain while the latter are quadratic or biquadratic representations in which the transition from one failure mode to another is gradual.

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A SURVEY OF FAILURE THEORIES OF ISOTROPIC AND ANISOTROPIC MATERIALS

R. S. SANDHU

I

FOREWORD

This work was conducted by Mr R. S. Sandhu, Exploratory Development Group, Advanced Composites Branch, at the Air Force Flight Dynamics Laboratory under project 4364, "Filamentary Composite Structures:, Task 436402, "Design Allowable and Criteria."

The manuscript was released by the author in January 1972. This technical report has been reviewed and approved.

Philip A. Parmley

PHILIP A. PARMLEY
Chief, Advanced Composites Branch
Structures Division

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INTRODUCTION

Filamentary composite materials are being used more and more in the construction of flight vehicles due mainly to their high strength and stiffness to weight ratios. They are formed by embedding fibers in matrix materials and are, therefore, nonhomogeneous and anisotropic. It is not expected that methods of analysis and design used for structures of homogeneous and isotropic materials, would be adequate for composites. However, they serve as the necessary background for the new techniques required for composite materials.

In structural application, strength is often a characteristic of interest. For homogeneous and isotropic materials a number of theories have been formulated from time to time to predict the response of the material under general state of stress using the material properties obtained from simple test conditions - uniaxial tension, compression and shear. These theories have served as the basis for the development of their counterparts for anisotropic materials. The generalization has been achieved by two approaches. In one approach, enough arbitrary parameters are introduced in the theory of strength for isotropic materials so that various failure modes can be explained. In the second development of the theory proceeds by assuming that the matrix is isotropic plastic material subjected to deformation constraints by the stiff fibers.

The aim of this report is to review the available theories of strength with emphasis on those for anisotropic materials. Section II is devoted to theories for isotropic materials while Section III deals with theories of anisotropic material.

SECTION II

THEORIES OF STRENGTH FOR ISOTROPIC MATERIALS

An accurate knowledge of strength characteristics of materials plays a vital role in an efficient design of structures. It enables a designer to determine the response of the structures under various conditions of loading. Generation of the material properties for the entire spectrum of loading is a very expensive procedure both in time and money. Instead information obtained from simple test conditions is generalized to more complex states, and the generalized hypothesis checked for selected load combinations, temperature, etc. Generalizations suggested by various investigators dating back to Galileo forming the subject matter of this Section are from References 1 to 7.

1. Galileo (1638)

Subjecting stones to simple tensile tests, he observed that the strength depended upon the cross sectional area of the bar and was independent of its length. He concluded that a fracture would occur when the "absolute resistance to fracture" (critical stress) was attained. He used this concept to investigate resistance to fracture of a cantilever beam but placed the neutral axis at the extreme compression fiber, which prevented him from obtaining correct results.

2. Coulomb (1773)

During Coulomb's period, stone was the principal material of construction. It was observed that stone specimens under uniaxial compression developed cracks which were inclined to the axis of loading. The inclination of these cracks differed from 45^{o} which is the inclination of the plane of maximum shearing stress under axial loading. This observation led

Coulomb to suggest that failure occurred when the shearing stress $|\tau|$ on the failure plane was equal to the sum of the cohesive strength 'c' and the frictional stress ($-\mu\ \sigma_n$) where μ is the coefficient of friction and σ_n is the normal tensile stress on the failure plane, i.e.,

$$|\tau| = c - \mu\sigma_n \tag{1}$$

On the basis of the Equation 1, it can be shown that the inclination of the failure plane is given by

$$\beta = 45^\circ - \phi/2 \tag{2}$$

where $\mu = \tan\phi$ (3)

Equation 1 implies that

(i) Compressive strength is greater than the tensile strength but the predicted ratio of strength is far less than found in practice;

(ii) The failure plane makes the same angle with the direction of the greatest principal stress for all states of stress including the one corresponding to pure tension (which is not true);

(iii) In the case of a three-dimensional stress field, failure is not affected by the intermediate principal stress.

However, in case of soils of low cohesion, Equation 1 yields reasonably good results.

3. Rankine, Lame', Clapeyron - Maximum Stress Theory (1858)

It is postulated in this theory that an element of a stressed body fails when the maximum principal stress attains a value obtained from a simple test like a uniaxial tension test, i.e.,

$$(\sigma_1^2 - \sigma_o^2)\ (\sigma_2^2 - \sigma_o^2)\ (\sigma_3^2 - \sigma_o^2) = 0 \tag{4}$$

where σ_1, σ_2, σ_3 are the principal stresses and σ_o is the failure stress in a simple test.

Equation 4 represents a cubic surface spaced symmetrically about the origin of coordinates. In case tensile and compressive strengths are different, the origin of coordinates is not symmetrically located. According to this theory if $\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_0$, $\sigma_3 = \sigma_0$ separately, failure is expected. However, simultaneous presence of $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_0$ does not cause failure of the material as shown by experiments. It produces a volume change only.

4. Saint Venant - Maximum Principal Strain Theory (1837)

This theory states that a failure (brittle or yielding) occurs under any state of stress when the maximum strain reaches the critical value obtained from simple tests, i.e.,

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \mu(\sigma_2 + \sigma_3)\right) \leq \frac{\sigma_0}{E} \tag{5}$$

where σ_1, σ_2, σ_3 are the principal stresses, ε_1 the strain, μ is the Poissons' ratio and E is the modulus of elasticity. Equation 5 yields

$$\frac{\sigma_1}{\sigma_0} \leq \frac{\mu}{\sigma_0}(\sigma_2 + \sigma_3) + 1 \tag{6}$$

For a hydrostatic state of stress $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, Equation 6 becomes

$$\sigma = \frac{\sigma_0}{(1-2\mu)} \tag{7}$$

Experiments indicate that the material can sustain much higher loads than indicated by Equation 7. In spite of its shortcomings, this theory held sway on the Continent for several years because of the support it received from St. Venant.

5. <u>Tresca - Maximum Shear Stress Theory (1864-1872)</u>

Observing the flow of soft metals in extrusion tests, Tresca stated that yielding in a material begins at a point where the maximum shear stress attains the yield stress value. In general form this criterion in terms of principal stress, σ_1, σ_2, σ_3 can be expressed as

$$\left[(\sigma_1-\sigma_3)^2 - \sigma_0^2\right]\left[(\sigma_2 - \sigma_1)^2 - \sigma_0^2\right]\left[(\sigma_3 - \sigma_2)^2 - \sigma_0^2\right] = 0 \qquad (8)$$

where σ_0 is the yield stress in tension test.

The Equation 8 represents three sets of parallel planes. Each set is normal to one coordinate plane. These planes define a hexagonal prism in σ_1, σ_2, and σ_3 stress space with its axis making equal angles with the coordinate axes.

It predicts satisfactorily the response of the metals such as mild steel under complex states of stress. It expects the slip lines to be inclined at 45^o to the major and minor principal stress directions.

In case of brittle materials which have different yield stresses in tension and compression, failure planes are different from the planes of maximum shear.

6. <u>Beltrami - Maximum Strain Energy Theory (1885)</u>

This theory postulates failure in a material when the total strain energy stored within the material reaches a value obtained from a simple test.

Strain energy 'U' per unit volume under a general state of stress is

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right] - \frac{\mu}{E}\left[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right] \tag{9}$$

Strain energy per unit volume in a simple tension test is given by the expression $\frac{\sigma_0^2}{2E}$.

By equating these expressions, Equation 10 is obtained.

$$\left(\frac{\sigma_1}{\sigma_0}\right)^2 + \left(\frac{\sigma_2}{\sigma_0}\right)^2 + \left(\frac{\sigma_3}{\sigma_0}\right)^2 - \frac{2\mu}{\sigma_0^2}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1\right) = 1 \tag{10}$$

The theory cannot be used as a criterion for the simple reason that under high hydrostatic pressure a large amount of energy may be stored without causing failure.

7. Mohr's Theory of Strength (1900)

Mohr devised a graphic representation of stress states and proved that states of stress on all planes intersecting the principal planes are represented by points lying in the shaded area (Figure 1). He used this representation of stress states to develop his strength theory. By this time there was enough evidence to indicate that shear stresses played an important role in determining the response of the material to the applied loading. Accordingly Mohr hypothesized that the plane which carries maximum shear stress is the weakest of all planes having the same normal stress. For example, of all the planes having the same normal stress OD (Figure 1), planes of maximum shear stresses (corresponding to the point D_2 and D_4 of the stress circle) are the weakest. This theory indicates that only the largest circle - the principal circle - of stress needs to be considered and the criterion becomes independent of σ_2.

If a sufficient number of principal circles, each related to the failure state of the material, are generated, envelopes (Figure 2) of these circles can be drawn and used to predict the failure state for any stress condition. The relationship between the shear stress τ and the normal stress σ on the plane of failure can be expressed as

$$|\tau| = F(\sigma) \tag{11}$$

where F (σ), a function of σ, is determined experimentally and is symmetric about the σ-axis.

One of the required characteristics of F (σ) is that it cannot have a negative root, as this would contradict the fact that materials under hydrostatic pressure do not deform plastically. Experiments by Von Karman indicated that for large negative values of σ, Mohr's envelope tends to bend in a direction parallel to the σ-axis.

Similarly two branches of the envelope intersecting at a finite slope at the positive point of σ-axis lead to anomalous results. For this reason, it was held for a long time that F (σ) had no physical significance as it approached the positive σ-axis. Alfons Leon (1935), however, pointed out that F (σ) had a tangible meaning if it intersected the positive σ-axis at a right angle and had a finite radius of curvature. Using this concept, it is possible to distinguish between cleavage and oblique fractures produced in a material under different stress states.

The simplest form of F (σ) is a pair of straight lines. As suggested by Mohr, they can be obtained by drawing outer tangents to the principal circles of stress corresponding to the failure states in tension and compression tests. In case of cast iron, the predicted strength in shear is $\frac{\sigma_t \sigma_c}{\sigma_t + \sigma_c}$ which agrees satisfactorily with the experimental value. In the simplified form, F (σ), Figure 3, can be written as

$$\tau = c - \mu\sigma \tag{12}$$

where c and μ are determined from the intercept which the tangent makes with the τ axis and its slope. In this form, it is the same as given by Equation (1). Equation 12 expressed in terms of principal stresses σ_1 and σ_3 becomes

$$\frac{\sigma_1}{S_t^1} - \frac{\sigma_3}{S_c} = 1 \tag{13}$$

where $$S_t^1 = \frac{2c \cos\phi}{1 + \sin\phi} \tag{14}$$

$$S_c = \frac{2c \cos\phi}{1 - \sin\phi} \tag{15}$$

Equation (13) can also be expressed as

$$(\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin\phi = 2c \cos\phi \tag{16}$$

which is similar to Equation (17) used by Guest to describe the test results on tension-torsion-pressure tubes.

$$(\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) C_1 = C \tag{17}$$

In case tensile and compressive strengths are equal, the angle ϕ is zero and Mohr's strength theory reduces to that of Tresca.

8. Distortional Energy Theory

This theory was re-discovered by several investigators before it was accepted as being valid for ductile metals. Noting that the hydrostatic state of stress produces only volume changes in the material, it is postulated that the material under a general state of stress yields when

(i) Its energy of distortion (Maxwell 1956, Huber 1904, Hencky 1924)

or

(ii) The second invariant, J_2, of the stress deviation (Von Mises 1913);

or

(iii) The mean square shear stress (Novozhilov 1952);

or

(iv) The octahedral shear stress (Nadai 1933);

attains a limiting value usually obtained from the simple tension test. All these are mathematically the same and lead to the expression of the criterion:

$$\frac{1}{2}\left[(\sigma_{zz} - \sigma_{yy})^2 + (\sigma_{zz} - \sigma_{xx})^2 + (\sigma_{xx} - \sigma_{yy})^2\right] + 3(\tau_{yz}^2 + \tau_{xz}^2 + \tau_{xy}^2) = \sigma_o^2 \qquad (18)$$

where σ's and τ's are the normal and shear stresses at a point and σ_o is the tensile yield stress.

In experimental verification, the plastic strain can be distinguished only when it becomes measurable. This measured amount of the plastic strain reflects more the mean square shear on all planes than the maximum shear stress. On this account, the experiments of Taylor and Quinney (1931) on copper, aluminum, and mild steel tubes favor Von Mises' criterion (Equation 18) more than Tresca's (T. H. Lin).

9. Pressure Dependent Failure Criterion

The yield condition for isotropic ductile materials is adequately represented in the stress space by Tresca's hexagonal or Von Mises' circular cylinder with its axis directed along the mean hydrostatic stress axis. In this failure condition, it is assumed that the materials have equal yield stresses in tension and compression and the yield surface is convex.

However, materials (cast iron, granular materials, concrete, natural rocks, etc) which are pressure sensitive cannot have a yield surface representation of a cylinder parallel to the hydrostatic axis. In general, they do not show sixfold symmetry of equipressure cross section for the simple reason that yield stresses of these materials in tension and in compression are not equal. For most of these materials Mohr's criterion with tension cut-offs is a reasonable first approximation, but when the effect of the intermediate principal stress becomes pronounced, it has to be generalized.

For idealized materials assuming that cross sections of the yield surfaces are geometrically similar, the yield surface could be expressed as a surface of revolution about the hydrostatic axis. Some of these generalizations are:

(i) Circular cone (Nadai), i.e.,

$$\tau_{oct}^2 = \frac{2}{9}\left[3\ C_o\ \sigma_{oct} - C_1\right]^2 \tag{19}$$

where τ_{oct} and σ_{oct} are octahedral shear and normal stresses and C_o & C_1 are the material constants;

(ii) Paraboloidal surfaces, i.e.,

$$(\sigma_1 + \sigma_2 + \sigma_3) = 3a_1 + 9a_2\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{20}$$

where σ_1, σ_2, σ_3 are the principal stresses and a_1, a_2 are the material constants; and

(iii) Criterion of Prager and Drucker

$$\alpha\ J_1 + \sqrt{J_2} = k \tag{21}$$

where J_1 is the first invariant of the stress tensor and J_2 is the second invariant of the stress deviation and α, k are material constants. None of these criteria, in simple forms or their generalizations, account for effects of temperature, time, environments, etc. No information is furnished about the behavior of the material subsequent to yielding in ductile materials or growth of the failure pattern in the brittle ones. The latter type of failure is associated with fracture of solids with almost no accompanying inelastic deformation. A study of these materials on microscale indicates that their strengths should be many orders more than obtained in tensile tests. To explain this phenomenon, Griffith (1921-24) proposed a theory that the energy required for fracture was not evenly distributed in the brittle solid. Due to the presence of randomly oriented cracks, high concentration of stresses occurred at their tips results in the uneven distribution of energy. This theory and later theoretical and experimental investigations about the formation and propagation of cracks is the subject matter which appropriately belong to "Fracture Mechanics" and will not be discussed in this report.

SECTION III

ANISOTROPIC THEORIES OF STRENGTH

Commonly used anisotropic materials are fibrous composites fabricated by embedding fibers in a suitable matrix material. This imparts orthotropic characteristics to the material. To extend the theories of strength of isotropic materials to composite materials, it is customary to treat them either as quasi-homogeneous or as a two-phase mixture of isotropic matrix subjected to geometric constraints by the fibers. In developing the theories, enough arbitrary parameters are introduced so that various failure modes can be incorporated. In some of the theories gradual transition from one failure mode to another is assumed. This gives rise to a smooth failure surface. On the other hand in certain formulations independence of failure modes is contemplated. These theories are broadly classified as theories with or without independent failure modes and are presented in this Section.

I. Theories with Independent Failure Modes

(1) Jenkins (1920)

Jenkins (References 8 and 9) extended the application of the maximum stress theory to a planar orthotropic material like wood to predict its failure. In this theory stresses acting on the orthotropic material are resolved along the material axes (σ_{11}, σ_{22}, τ_{12}) and it is postulated that the failure will occur when one or all of (σ_{11}, σ_{22}, τ_{12}) attain the maximum values X, Y, S obtained by simple loading conditions, i.e., the failure is precipitated when any one of the following conditions is satisfied:

$$\left.\begin{aligned}\sigma_{11} &= X \\ \sigma_{22} &= Y \\ \tau_{12} &= S\end{aligned}\right\} \qquad (22)$$

where X and Y could be tensile or compressive stresses at failure.

(2) Stowell and Liu (1961)

In this hypothesis (Reference 10) three failure modes are recognized.

(i) Brittle fracture of fibers;

(ii) Yielding of matrix in shear parallel to fibers; and

(iii) Yielding of matrix in tension transverse to the fibers.

The failure of the material occurs when σ_{11}, σ_{22}, σ_{12}, normal and shear stresses along the material axes attain limiting values, ie.e, when

$$\sigma_{11} = X_f \qquad (23)$$

$$\sigma_{22} = Y_m \qquad (24)$$

$$\tau_{12} = S_m \qquad (25)$$

where X_f is the failure stress of fibers, Y_m and S_m are the tensile and shear strengths of the matrix.

Kelly and Davis (Reference 11) pointed out that the use of the bulk properties of the matrix in Equation 24 and 25 ignored the observed fiber matrix interaction in composites. It was suggested that Y_m and S_m be increased by factors of 1.15 and 1.5 respectively to incorporate the interaction effect.

Experiments of Jackson and Cratchley (Reference 12) to correlate the strength and the mode of failure with the fiber orientation of unidirectional and angle ply laminates led them to observe:

(i) Using experimentally determined values of Y_m and S_m, Equations 23 to 25 are quite adequate to predict the behavior of off-axis specimens of unidirectional laminates, particularly when the change in angle ($\sim 3.5^o$) in testing is allowed for;

(ii) No strength peak is noticed when the mode of failure changes from that of shear to the transverse one with the increasing orientation angle; and

(iii) Equations 23 - 25 do not describe adequately the performance of angle ply laminates.

Further studies of the Stowel and Liu criterion were made by Cooper (Reference 13). Thin sheets, in which tungsten wires were weakly bonded to the copper matrix, were used to fabricate off-axis coupons. The experimental results were in reasonable agreement with the theory except for two zones of deviations, namely, when the angle of orientation ϕ was:

(a) $0^o < \phi < 10^o$

(b) $45^o < \phi < 90^o$

The discrepancy in the case (a) is attributed to the fibers terminating at the free edge being unstressed thereby reducing the effective width of the specimen. This edge effect diminishes rapidly with increasing ϕ due to the decreasing overall fracture stress and the component of stress along the fibers.

In the second case (b) contrary to the theory, no increase of strength was observed. In fact it slowly decreased. The explanation offered is that it is always possible to find a plane inclined at 45^{o} to the loading axis that does not cut any of the fibers.

(3) Prager (1969)

The Reference 14 states that the second and the third failure mechanisms (Equations 24 and 25) representing the failure by plastic flow of the matrix are not independent of each other, but interact. This interaction is established by assuming the matrix is a homogeneous, perfectly plastic solid obeying the yield condition and flow rule of Von Mises subject to the constraint of vanishing rate of extension in the direction of indefinitely thin reinforcing fibers. These assumptions are used to develop a set of equations governing the behavior of unidirectional and angle ply laminates. The stresses acting on the laminates are the principal ones.

Unidirectional laminates

$$\sigma_x = \frac{4\,\sigma_{11}{}^{F}}{\left(3(1-c)\cos 2\theta + (1+c)\right)} \tag{26}$$

or

$$\sigma_x = \frac{4k}{\sqrt{\left[(1+c) - (1-c)\cos 2\theta\right]^2 + 4(1-c)^2 \sin^2 2\theta}} \tag{27}$$

where σ_x, $C\sigma_x$ = Principal stresses acting on the laminate

$\sigma_{11}{}^{F}$ = Fiber stress at failure

[illegible] = Yield shear [illegible] of the matrix

and θ = [illegible]ngle which σ_x makes with the [illegible]ber direction

Using Equations (26 and 27), the ranges of fiber failure and matrix failure modes can be determined.

(4) Waddoups (1966)

In the application of St Venant's maximum strain theory to anisotropic materials (Reference 15), it is hypothesized that the failure is precipitated when any of the strain components associated with material axes attains the limiting value. If $\bar{\varepsilon}_{11}$, $\bar{\varepsilon}_{22}$, $\bar{\varepsilon}_{12}$ are the limiting strain values in a planar orthotropic material, the conditons of failure in terms of stresses are expressed as

(i) $$\sigma_{22} = -\frac{1}{\mu_{12}}(P_{11} - \sigma_{11}) \qquad (31)$$

(ii) $$\sigma_{22} = P_{22} + \mu_{12}\frac{E_{22}}{E_{11}}\sigma_{11} \qquad (32)$$

(iii) $$\tau_{12} = P_{12} \qquad (33)$$

where $$P_{11} = \bar{\varepsilon}_{11} E_{11} \qquad (34)$$

$$P_{22} = \bar{\varepsilon}_{22} E_{22} \qquad (35)$$

$$P_{12} = \bar{\gamma}_{12} G_{12} \qquad (36$$

and E_{11}, E_{22}, G_{12} are the moduli of elasticity of the material. Equations 31, 32, and 33 define longitudinal, transverse, and shear modes of failure.

(5) Lance and Robinson (1971)

The first attempt to use the maximum shear stress (Tresca) criterion for anisotropic materials was by Hu (Reference 16). In developing the theory it is assumed that material axes of orthotropy coincide with the principal stress directions. An extension of this by Wasti (Reference 17) takes into account the effect of the presence of reinforcement. A general maximum shear stress based yield theory, however, is developed in Reference 18 for a composite material consisting of stiff parallel ductile fibers embedded in ductile matrix.

Yielding of the composite is assumed to occur when the maximum shear stress on

(i) Planes parallel to the fibers and acting in a perpendicular direction; or

(ii) Planes parallel to the fibers and acting in the same direction as fibers; or

(iii) Planes oriented at 45° to the fiber direction; attains a critical value of stress associated with the failure planes.

In Figure 4, X coincides with the fiber direction, $\bar{a}$, $\bar{t}$, $\bar{n}$ are the unit vectors. To meet the requirements of assumptions (i) and (ii), the components of shear stress on the plane to which $\bar{n}$ (Figure 4) is normal and acting in the directions $\bar{a}$ and $\bar{t}$ should satisfy the conditions:

$$\text{(i)} \quad \sigma_{yx} \text{ Cos } \phi + \sigma_{zx} \text{ Sin } \phi = K_a \tag{37}$$

$$\text{(ii)} \quad 1/2\, (\sigma_{zz} - \sigma_{yy}) \text{ Sin } 2\phi + \sigma_{yz} \text{ Cos } 2\phi = K_t \tag{38}$$

where K_a and K_t are critical values of shear stress in $\bar{a}$ and $\bar{t}$ direction.

The expressions in Equations 37 and 38 attain the maximum values when

$$\phi_a = \tan^{-1} \left(\frac{\sigma_{zx}}{\sigma_{yz}}\right) \tag{39}$$

$$\phi_t = 1/2 \tan^{-1} \left[(\sigma_{zz} - \sigma_{yy})/2\sigma_{yz} \right] \tag{40}$$

In Figure 5, $\vec{n}$ is normal unit vector to the plane of failure. The angle ψ is the angle between of $\vec{n}$ on ZY plane and Z-axis. To satisfy the condition (iii),

$$1/2 \left| \sigma_{xx} - \sigma_{yy} \, Cos^2\psi - {}_{zz} \, Sin^2\psi - \sigma_{yz} \, Sin^2\psi \right| = K_s \quad (41$$

where K_s is the critical shear stress on planes inclined at 45^0 to the X-axis. The left hand side of Equation 41 is maximized for

$$\psi_s = 1/2 \tan^{-1} \left[2\sigma_{yz} / (\sigma_{yy} - \sigma_{zz}) \right] \quad (42)$$

In case of a thin sheet of composite material in a state of plane stress, Figure 6,

$$\sigma_{zz} = \sigma_{yz} = \sigma_{xz} = 0 \quad (43)$$

The other components of stress are

$$\sigma_{yy} = 1/2 \left[\sigma_{11} + \sigma_{22} \right] - 1/2 \left[\sigma_{11} - \sigma_{22} \right] Cos \; 2 \; \theta \quad (44)$$

$$\sigma_{xx} = 1/2 \left[\sigma_{11} + \sigma_{22} \right] + 1/2 \left[\sigma_{11} - \sigma_{22} \right] Cos \; 2 \; \theta \quad (45)$$

$$\sigma_{xy} = 1/2 \; (\sigma_{11} - \sigma_{22}) \; Sin \; 2 \; \theta \quad (46)$$

Critical shear planes corresponding to the stresses (Equations 43 to 46) are

$$\phi_a = \frac{\pi}{2}; \; \phi_t = \frac{\pi}{4}, \frac{3\pi}{4}; \; \phi_s = 0, \frac{\pi}{2} \quad (47)$$

Using the results of Equations 43 to 47, Equations 37, 38, and 41 become

$$|(\sigma_{11} - \sigma_{22}) \, \mathrm{Sin}\, 2\,\theta| = 2\, Ka \tag{48}$$

$$|\sigma_{11} \, \mathrm{Sin}^2\theta + \sigma_{22} \, \mathrm{Cos}^2\theta| = 2\, Kt \tag{49}$$

$$|\sigma_{11} \, \mathrm{Cos}^2\theta + \sigma_{22} \, \mathrm{Sin}^2\theta| = 2\, Ks \tag{50}$$

$$|(\sigma_{11} - \sigma_{22}) \, \mathrm{Cos} 2\theta| = 2\, Ks \tag{51}$$

For uniaxial stress field σ_{11}, Equations 48 - 51 become

$$\sigma_{11} = \begin{cases} 2\ Ka/\mathrm{Sin}\ 2\ \theta & (52) \\ 2\ Kt/\mathrm{Sin}^2\theta & (53) \\ 2\ Ks/\mathrm{Cos}^2\theta & (54) \\ 2\ Ks/\mathrm{Cos} 2\ \theta & (55) \end{cases}$$

No interaction between the shear modes is allowed for in the criterion.

(II) <u>THEORIES WITHOUT INDEPENDENT FAILURE MODES</u>

(1) <u>Hill (1948)</u>

In metals, grains are randomly oriented. It imparts isotropic properties to metals on macroscale. This isotropic behavior is confined to small deformations. With increasing strains, grains align themselves in preferred directions. It renders the material anisotropic.

Hill (References 19, 20) suggested a yield criterion that allows for the anisotropy. In the criterion it is assumed that the material possesses three orthogonal planes of symmetry and there is no Bauschinger effect. It is of the form

$$2f(\sigma) = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2 + 2L\ \tau_{23}^2 + 2M\ \tau_{13}^2 + 2N\ \tau_{12}^2 = 1 \tag{56}$$

where F, G, H, L, M, N are the parameters characteristic of the current state of anisotropy. If X, Y, Z are tensile yield stresses and R, T, S are the yield stresses in shear with respect to the axes of anisotropy, Equation 56 can be expressed as

$$(\frac{\sigma_{11}}{X})^2 + (\frac{\sigma_{22}}{Y})^2 + (\frac{\sigma_{33}}{Z})^2 - \left[(\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2})\sigma_{11}\sigma_{22} + (\frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2})\sigma_{22}\sigma_{33} + (\frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2})\sigma_{11}\sigma_{33}\right] + (\frac{\tau_{12}}{S})^2 + (\frac{\tau_{23}}{T})^2 + (\frac{\tau_{13}}{R})^2 = 1 \tag{57}$$

In case of isotropic materials,

$$X = Y = Z = \sigma_o, \text{ and } T = R = S = \frac{\sigma_o}{\sqrt{3}} \tag{58}$$

Equation 57 reduces to Von Mises' criterion. For generalized plane stress state, Equation 57 becomes

$$(\frac{\sigma_{11}}{X})^2 + (\frac{\sigma_{22}}{Y})^2 - (\frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2})\sigma_{11}\sigma_{22} + (\frac{\tau_{12}}{S})^2 = 1 \tag{59}$$

(2) Marin (1956)

Marin (Reference 21) modified the distortion energy criterion for orthotropic materials with different strengths in tension and compression and expressed it as

$$(\sigma_{11} - a)^2 + (\sigma_{22} - b)^2 + (\sigma_{33} - c)^2 + q\left[(\sigma_{11} - a)(\sigma_{22} - b) + (\sigma_{22} - b)(\sigma_{33} - c) + (\sigma_{33} - c)(\sigma_{11} - a)\right] = \sigma^2 \tag{60}$$

where a, b, c, etc are to be determined experimentally. For plane stress condition Equation 60 can be expressed as

$$\sigma_{11}^2 + K_1 \sigma_{11} \sigma_{22} + \sigma_{22}^2 + K_2 \sigma_{11} + K_3 \sigma_{22} = K_4 \tag{61}$$

where K_1, K_2, K_3, and K_4 are constants. They can be evaluated from tests

$$\left.\begin{array}{lllll} \text{(i)} & \sigma_{11} = X_T & & \sigma_{22} = o \\ \text{(ii)} & \sigma_{22} = Y_T & & \sigma_{11} = o \\ \text{(iii)} & \sigma_{11} = X_C & & \sigma_{22} = o \\ \text{(iv)} & \sigma_{11} = -\sigma_{22} = S & & \end{array}\right] \tag{62}$$

Substitution of Equations 62 in Equation 61 yields

$$K_1 = 2 - \frac{X_c X_T - S\left(X_c - X_T - X_c \left(\frac{X_T}{Y_T}\right) + Y_T\right)}{S^2} \tag{63}$$

$$K_2 = X_c - X_T \tag{64}$$

$$K_3 = X_c X_T / Y_T - Y_T \tag{65}$$

$$K_4 = X_T Y_c \tag{66}$$

In the formulation of this criterion, it is assumed that the principal stress directions coincide with the material axes, therefore, there is no way to account for shear stresses that may be present along the material axes.

(3) Stassi D'Alia (1959).

Reference 22 gives the relationship between the stresses and the strengths of the material as

$$\frac{\sigma_{11}^2 - \sigma_{11} \sigma_{22} + \sigma_{22}^2}{X_T X_c} + \left(1 - \frac{X_T}{X_c}\right) \frac{\sigma_{11} + \sigma_{22}}{X_T} + 3\left(\frac{\tau_{12}}{X_T}\right)^2 = 1 \tag{67}$$

In Equation 67 shear strength is not an independent quantity but is a function of X_T. This criterion assumes the material to be isotropic with different properties in tension and compression and, therefore, it is not valid for application to the composite materials.

(4) Norris (1962)

Using nine strength properties, namely three tensile, three compressive, and three shear strengths, associated with material axes of the orthotropic material, Norris (Reference 23) suggested an interaction type relationship between stresses and strengths. The failure of the material would occur if any one of the following equations is satisfied.

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{XY} + \left(\frac{\tau_{12}}{S}\right)^2 = 1 \tag{68}$$

$$\left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\sigma_{33}}{Z}\right)^2 - \frac{\sigma_{22}\sigma_{33}}{YZ} + \left(\frac{\tau_{23}}{T}\right)^2 = 1 \tag{69}$$

$$\left(\frac{\sigma_{33}}{Z}\right)^2 + \left(\frac{\sigma_{11}}{X}\right)^2 - \frac{\sigma_{33}\sigma_{11}}{ZX} + \left(\frac{\tau_{13}}{R}\right)^2 = 1 \tag{70}$$

where X, Y, and Z can be tensile or compressive consistent with the nature of the stresses σ_{11}, σ_{22}, and σ_{33}.

Equations 68 to 70 for the generalized plane stress condition become:

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 - \frac{\sigma_{11}\sigma_{22}}{XY} + \left(\frac{\tau_{12}}{S}\right)^2 = 1 \tag{71}$$

$$\left(\frac{\sigma_{22}}{Y}\right)^2 = 1 \tag{72}$$

$$\left(\frac{\sigma_{11}}{X}\right)^2 = 1 \tag{73}$$

In Reference 25 it is stated that the shear strength S was not determined but deduced from Equations 71 to 73 by using tension data and the results made to fit the formulation.

(5) Griffith and Baldwin (1962)

Defining $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_2$, $\sigma_{33} = \sigma_3$, $\tau_{23} = \sigma_4$, $\tau_{13} = \sigma_5$, $\tau_{12} = \sigma_6$, and a set of strain components ε_{ij} in the same fashion, stress-strain relations of an anisotropic linearly elastic solid are expressed as

$$\sigma_i = C_{ij}\,\varepsilon_j \quad (i,j = 1,\dots 6) \tag{74}$$

$$\varepsilon_i = S_{ij}\,\sigma_j \quad (i,j = 1,\dots 6) \tag{75}$$

Total strain energy UT per unit volume for orthotropic materials (Reference 24) is given by

$$U_T = \int_{\varepsilon} \sigma_i \, d\,\varepsilon_i \tag{76}$$

$$= 1/2\left(S_{11}\,\sigma_1^2 + S_{22}\,\sigma_2^2 + S_{33}\,\sigma_3^2\right) + \left(S_{12}\,\sigma_1\,\sigma_2 + S_{13}\,\sigma_1\,\sigma_3 + S_{23}\,\sigma_2\,\sigma_3\right) + \left(S_{44}\,\sigma_4^2 + S_{55}\,\sigma_5^2 + S_{66}\,\sigma_6^2\right) \tag{77}$$

The corresponding volumetric strain energy Uv is

$$Uv = (\sigma_1 + \sigma_2 + \sigma_3)\,\frac{\varepsilon_\nu}{3} \tag{78}$$

where $\varepsilon_\nu = \left(\sigma_1(S_{11} + S_{12} + S_{13}) + \sigma_2\,(S_{12} + S_{22} + S_{23}) + \sigma_3(S_{13} + S_{23} + S_{33})\right)/3$ (79)

The distortional energy U_D for the orthotropic material can now be written as

$$U_D = U_T - U_v = \frac{\sigma_1^2}{3}\left[S_{11} - \frac{S_{12} + S_{13}}{2}\right] + \frac{\sigma_2^2}{3}\left[S_{22} - \frac{S_{12} + S_{23}}{2}\right] +$$

$$\frac{\sigma_3^2}{3}\left[S_{33} - \frac{S_{13} + S_{23}}{2}\right] + \frac{\sigma_1\sigma_2}{3}\left[2S_{12} - \right.$$

$$\left.\frac{S_{11} + S_{22} + S_{13} + S_{23}}{2}\right] + \frac{\sigma_2\sigma_3}{3}\left[2S_{23} - \right. \tag{80}$$

$$\left.\frac{S_{12} + S_{22} + S_{13} + S_{33}}{2}\right] + \frac{\sigma_1\sigma_3}{3}\left[2S_{13} - \right.$$

$$\left.\frac{S_{11} + S_{12} + S_{23} + S_{33}}{2}\right] +$$

$$S_{44}\sigma_4^2 + S_{55}\sigma_5^2 + S_{66}\sigma_6^2$$

For plane stress condition Equation 80 reduces to

$$U_D = \frac{\sigma_1^2}{3}\left[S_{11} - \frac{S_{12} + S_{13}}{2}\right] + \frac{\sigma_2^2}{3}\left[S_{22} - \frac{S_{12} + S_{23}}{2}\right] +$$

$$\frac{\sigma_1\sigma_2}{3}\left[2S_{12} - \frac{S_{11} + S_{22} + S_{13} + S_{23}}{2}\right] + S_{66}\sigma_6^2 \tag{81}$$

It is assumed in this hypothesis that the incipient yielding or the fracture will occur when U_D equals a critical amount of available distortion energy U_{D_c} . U_{D_c} is obtained by a single uniaxial test with known orientation of the principal axis of orthotropy relative to the loading.

Results of this one parameter theory are shown to be in fair agreement with the experimental data. However, it is suggested by Griffith and Baldwin that additional tests under more general states of stress are required before accepting the theory based on the distortional energy.

(6) Azzi and Tsai (1965)

Azzi and Tsai (Reference 25) assume that fiber reinforced composites are transversally isotropic, i.e.,

$$Z = Y \tag{82}$$

Using the relation from Equation 82 in Equation 59, the modified form of Hill's criterion becomes

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 - \frac{\sigma_{11}\,\sigma_{22}}{X^2} + \left(\frac{\tau_{12}}{S}\right)^2 = 1 \tag{83}$$

Results of tests on off-axis coupons prepared from unidirectional glass-epoxy laminates are shown to be in agreement with the predictions based on Equation 83.

Later (Reference 41), Equation 83 is generalized to account for different tensile and compressive strengths by stating that X and Y should be consistant with the nature of stresses σ_{11} and σ_{22}.

(7) Goldenblat and Kopnov (1965)

Reference 26 marks the first attempt to develop a general strength theory for anisotropic material. It lists some basic requirements which the theory must fulfill. They are:

(i) It must be capable of predicting the strength of the material under complex states of stress for which no experimental data is available;

(ii) Besides having the stress tensor characterizing the stress state, the expression for strength should incorporate in some way the strength properties of the material;

(iii) It should include transition from one coordinate system to another including simple state of stress in any coordinate system;

(iv) It should have the structure of an invariant formed from stress tensor and strain tensor components characterizing the strength properties of the material;

(v) The relationships between material constants should be independent of the coordinate system;

(vi) It should have a capability to account for difference in tensile and compressive strengths which results in shear strength being dependent upon the sign of the tangential stresses;

(vii) The failure surface is assumed to satisfy a heuristic principle which asserts that the growth of strength indices is possible only when the new failure surfaces include the old ones so that there is no intersection of the surfaces.

The failure criterion satisfying the conditions (i) to (vii) can be written as:

$$(F_{ik}\,\sigma_{ik})^{\alpha} + (F_{pqnm}\,\sigma_{pq}\,\sigma_{nm})^{\beta} + (F_{rstlmn}\,\sigma_{rs}\,\sigma_{tl}\,\sigma_{nm})^{\gamma} + \ldots \leq 1 \qquad (84)$$

Equation 84 reduces to the strength of material formulas $\alpha = 1$, $\beta = 1/2$ and $\gamma = 1/3$ etc.

Selecting the first two terms of the expansion, the criterion becomes

$$F_{ik}\,\sigma_{ik} + \sqrt{F_{pqrs}\,\sigma_{pq}\,\sigma_{rs}} \leq 1 \quad (i,k,p,q,r,s = 1,\ldots 3) \qquad (85)$$

The comment made in Reference 26 is that the final selection of the expression depends upon its being verified by experiments. If the results of experiments require, more terms can be included. The strength tensors F_{ik} and F_{pqrs} satisfy the following symmetry conditions:

$$F_{ik} = F_{ki};\ F_{pqrs} = F_{qprs};\ F_{pqrs} = F_{pqsr};\ F_{pqrs} = F_{rspq} \tag{86}$$

Using the symmetry conditions, and condensing the tensors of stress and strength, Equation 85 can be written as

$$F_i\,\sigma_i + \sqrt{F_{ij}\,\sigma_i\,\sigma_j} \le 1 \qquad (i,j = 1,2\ldots6) \tag{87}$$

If $F_i^{\bullet}$ and $F_{ij}^{\bullet}$ are the parameters of strength in fundamental coordinates, i.e., a system pertaining to the material axes of the orthotropic body, Equation (87) becomes

$$F_i^{o}\,\sigma_i + \sqrt{F_{ij}^{o}\,\sigma_i\,\sigma_j} \le 1 \tag{88}$$

where F_o^{o} and F_{ij}^{o} are evaluated from test conditions.

(i) Stress conditions of pure tension and compression along the material axes, yield

$$F_1^{o} = 1/2\left(\frac{1}{X_T} - \frac{1}{X_C}\right);\ F_{11}^{o} = 1/4\left(\frac{1}{X_T} + \frac{1}{X_C}\right)^2 \tag{89}$$

$$F_2^{o} = 1/2\left(\frac{1}{Y_T} - \frac{1}{Y_C}\right);\ F_{22}^{o} = 1/4\left(\frac{1}{Y_T} + \frac{1}{Y_C}\right)^2 \tag{90}$$

$$F_3^{o} = 1/2\left(\frac{1}{Z_T} - \frac{1}{Z_C}\right);\ F_{33}^{o} = 1/4\left(\frac{1}{Z_T} + \frac{1}{Z_C}\right)^2 \tag{91}$$

(ii) Stress conditions (Figures 7a and 7b)

$$\sigma_1 = \tau_6\,(45)^+,\ \sigma_2 = -\,\tau_6(45)^+$$

$$\sigma_1 = -\,\tau_6(45)^-,\ \sigma_2 = \tau_6(45)^-$$

and similar conditions for other planes yield

$$F^o_{12} = 1/8 \left[(\frac{1}{X_T} + \frac{1}{X_C})^2 + (\frac{1}{Y_T} + \frac{1}{Y_C})^2 - (\frac{1}{\tau_{6(45)^+}}) + \frac{1}{\tau_{6(45)^-}})^2 \right] \quad (92)$$

$$F^o_{13} = 1/8 \left[(\frac{1}{X_T} + \frac{1}{X_C})^2 + (\frac{1}{Z_T} + \frac{1}{Z_C})^2 - (\frac{1}{\tau_{4(45)^+}} + \frac{1}{\tau_{4(45)^-}})^2 \right] \quad (93)$$

$$F^o_{23} = 1/8 \left[(\frac{1}{Y_T} + \frac{1}{X_C})^2 + (\frac{1}{Z_T} + \frac{1}{Z_C})^2 - (\frac{1}{\tau_{5(45)^+}} + \frac{1}{\tau_{5(45)^-}})^2 \right] \quad (94)$$

To evaluate F^o_{12}, F^o_{13}, F^o_{23}, two tests each have been employed. To remove any contradiction resulting therefrom, additional conditions required are:

$$\left[\frac{1}{X_T} - \frac{1}{Y_T} - \frac{1}{X_C} + \frac{1}{Y_C} \right] = \frac{1}{\tau_{6(45)^+}} - \frac{1}{\tau_{6(45)^-}} \quad (95)$$

$$\left[\frac{1}{X_T} - \frac{1}{Z_T} - \frac{1}{X_C} + \frac{1}{Z_C} \right] = \frac{1}{\tau_{4(45)^+}} - \frac{1}{\tau_{4(45)^-}} \quad (96)$$

$$\left[\frac{1}{Y_T} - \frac{1}{Z_Y} - \frac{1}{Y_C} \quad \frac{1}{Z_C} \right] = \frac{1}{\tau_{5(45)^+}} - \frac{1}{\tau_{5(45)^-}} \quad (97)$$

(iii) Conditions of pure shear (Figure 8)

$\tau = \sigma_6(0)^+$, and $\tau = \sigma_6(0)^-$ etc.

yield

$$F^o_{66} = 1/4 \left[\frac{1}{\sigma_{6(0)^+}} \right]^2 \quad (98)$$

$$F^o_{44} = 1/4 \left[\frac{1}{\sigma_{4(0)^+}} \right]^2 \quad (99)$$

$$F^o_{55} = 1/4 \left[\frac{1}{\sigma_{5(0)^+}} \right]^2 \quad (100)$$

and F^o_4, F^o_5, F^o_6 are zero in fundamental cooıdinates as

$\sigma_{4(0)^+} = \sigma_{4(0)^-}$; $\sigma_{5(0)^+} = \sigma_{5(0)^-}$; and $\sigma_{6(0)^+} = \sigma_{6(0)^-}$.

All other F's are zero in fundamental coordinates as there is no coupling between shearing stresses and the normal stresses in that coordinate system.

Having computed F_i^o and F_{ij}^o in the fundamental system, F_i, F_{ij} in any other coordinate system can be determined by transformation.

In case of generalized plane stress condition in fundamental system, strength criterion becomes

$$F_1^o \sigma_1 + F_2^0 \sigma_2 + \sqrt{F_{11}^o \sigma_1^2 + 2F_{12}^o \sigma_1\sigma_2 + F_{22}^o \sigma_2^2 + F_{66}^o \sigma_6^2} \leq 1 \qquad (101)$$

Experiments on tubular specimens of fiber glass under biaxial states of stress were found to yield satisfactory results (References 26,27).

(8) <u>Ashkenazi (1965)</u>

To formulate a theory of failure of orthotorpic materials, Ashkenazi (References 28, 29, 30, 31) assumes that

(i) The material is uniform, continuous, compact and anisotropic;

(ii) The factors like time, temperature, humidity, specimen size and shape, etc., can be ignored:

(iii) The strength properties are tensorial in character and can be represented by a tensor of the fourth order, i.e.,

$$a_{i'k'o'p'} = \alpha^1_{i'i} \alpha^2_{k'k} \alpha^3_{o'o} \alpha^4_{p'p} a_{ikop} \qquad (102)$$

where (i', i, k', k, o', o, p', p = 1,...3) (103)

in cartesian coordinates; and (iv) it is postulated that

$$a_{ikop} = \frac{1}{\sigma_{ikop}} \qquad (104)$$

where σ_{1kop} is the strength characteristic of the material.

On the basis of these assumptions, the author derives expressions to determine the normal and shear stresses which the orthotropic material can sustain in any arbitrary direction. These expressions in the case of a plane problem reduces to

$$\frac{1}{\sigma_b} = \frac{\cos^4\alpha}{X} + \left(\frac{4}{X^{45}} - \frac{1}{X} - \frac{1}{Y}\right) \sin^2\alpha \cos^2\alpha + \frac{\sin^4\alpha}{Y} \qquad (105)$$

$$\frac{1}{\tau_b} = \frac{\cos^2 2\alpha}{S} + \frac{4 \sin^2\alpha \cos^2\alpha}{S^{45}} \qquad (106)$$

where

σ_b, τ_b = normal and shear stresses in x', y' coordinate system making an angle α with the material axes.

X^{45}, S^{45} = normal and shear strength obtained from a coupon cut at 45° to the material axes.

Equations 104 and 105 can be written as

$$\sigma_b = \frac{X}{\cos^4\alpha + b \sin^2 2\alpha + C \sin^4\alpha} \qquad (107)$$

$$\tau_b = \frac{1}{\frac{\cos^2 2\alpha}{S} \quad \frac{\sin^2 2\alpha}{S^{45}}} \qquad (108)$$

$$\text{when} \quad b = \frac{X}{X^{45}} - \frac{1+C}{4}$$

$$\text{and} \quad C = \frac{X}{Y} \qquad (109)$$

In case a sixth order tensor is used to represent the strength properties of the material, Equation 106 becomes

$$\sigma_b = \frac{\sigma_o}{\cos^6\alpha + b_o \sin^2 2\alpha + C \sin^6\alpha}$$

where $C = \frac{X}{Y}$

and $b_o = \frac{X}{X^{45}} - \frac{1 + C}{8}$ (110)

Experimental results appear to agree with the calculated values using Equation 106. The use of Equation 109 instead of Equation 106 does not yield any more satisfactory results.

The Equations 106 and 107 are applicable only if σ_b or τ_b alone is acting on the element of the material. For complex states of stress, Ashkenazi considers the possibility of using Von Mises' plasticity function of the form

$$J_2 = F_{ikop}\,\sigma_{ik}\,\sigma_{op} = \text{constant} \tag{111}$$

which represents a surface in stress space.

For the plane stress state, Equation 111 becomes

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 + 2F_{12}\,\sigma_{11}\,\sigma_{22} = 1 \tag{112}$$

In case the stress state is given by

$$\sigma_{11} = \sigma_{22} = \tau_{12} = \frac{X^{45}}{2} \tag{113}$$

the value of F_{12} evaluated from Equation 112 is

$$2F_{12} = \left[\frac{4}{(X^{45})^2} - \frac{1}{X^2} - \frac{1}{Y^2} - \frac{1}{S^2}\right] \tag{114}$$

Using Equation 114, Equation 112 can be written as

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 + \left[\frac{4}{(X^{45})^2} - \frac{1}{X^2} - \frac{1}{Y^2} - \frac{1}{S^2}\right]$$

$$\sigma_{11}\,\sigma_{22} = 1 \tag{115}$$

For highly anisotropic material the unconstrained value of F_{12} gives rise to discontinuous curve for off-axis properties at certain angles. Equation 106 does not appear to have this drawback. For a weak anisotropic material, an algebraic equation of second degree of the form of Equation 111 or Equation 115 can adequately describe the behavior under complex states of stress. In case the failure surfaces have certain portions which are concave, the second degree equations are not sufficient. A fourth order polynomial can be used with advantage and Ashkenazi suggests the fourth order polynomial of the form

$$
\begin{aligned}
&\left[\frac{\sigma_{11}^2}{X} + \frac{\sigma_{22}^2}{Y} + \sigma_{11}\sigma_{22}\left(\frac{4}{X_{45}} - \frac{1}{X} - \frac{1}{Y} - \frac{1}{S}\right) + \frac{\tau_{12}^2}{S}\right]^2 \\
&+ 2\,\frac{\sigma_{11}\sigma_{22} - \tau_{12}^2}{S}\left[\sigma_{11}\sigma_{22}\left(\frac{1}{X} + \frac{1}{Y}\right) + \frac{\sigma_{11}^2}{X} + \frac{\sigma_{22}^2}{Y}\right] - \\
&(\sigma_{11}\sigma_{22} - \tau_{12}^2)\left[\sigma_{11}\sigma_{22}(\lambda + \mu) + \lambda\sigma_{11}^2 + \mu\sigma_{22}^2\right] + \\
&\rho(\sigma_{11}\sigma_{22} - \tau_{12}^2)(\sigma_{11} + \sigma_{22}) - (\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{11}\sigma_{22} + \tau_{12}^2) = 0
\end{aligned}
\qquad (116)
$$

when λ, μ, and ρ are to be determined experimentally for three biaxial stress states.

Equation 116 includes Equations 106 and 107 as particular cases.

(9) <u>Malmeister (1965)</u>

The ultimate resistance of materials can be determined possibly by using stresses, strains or the energy expended to reach the failure states. Malmeister (Reference 32) bases the failure criterion on the stress states and postulates the failure to occur when the stress path represented by the ray from the origin of coordinates of stress space terminates on the failure surface given by

$$F_{\alpha\beta}\,\sigma_{\alpha\beta} + F_{\alpha\beta\gamma\delta}\,\sigma_{\alpha\beta}\,\sigma_{\gamma\delta} + \;\cdots\cdots = 1 \qquad (117)$$

where $F_{\alpha\beta}$, $F_{\alpha\beta\gamma\delta}$, etc.are the strength tensors of second, fourth and higher order.

In case of plane stress conditions, Equation 117 with two terms only can be written as

$$F_{xx}\,\sigma_{xx} + F_{yy}\,\sigma_{yy} + F_{xy}\,\tau_{xy} + F_{xxxx}\,\sigma_{xx}^2 + F_{yyyy}\,\sigma_{yy}^2 +$$

$$F_{xyxy}\,\tau_{xy}^2 + 2(F_{xxyy}\,\sigma_{xx}\,\sigma_{yy} + F_{xxxy}\,\sigma_{xx}\,\tau_{xy} + F_{yyyx}\,\sigma_{yy}\,\tau_{xy}) = 1 \quad (118)$$

when x, y are arbitrarily located cartesian coordinates. Equation 118 has nine coefficients which need to be determined from nine tests, namely two tensile, two compressive, one shear, and four biaxial tests. Equation 118 is considerably simplified for orthotropic material. On condensing the strength and stress tensors Equation 118 related to material axes becomes

$$F_1\,\sigma_1 + F_2\,\sigma_2 + F_6\,\sigma_6 + F_{11}\,\sigma_1^2 + F_{22}\,\sigma_2^2 + F_{66}\,\sigma_6^2 + 2F_{12}\,\sigma_1\,\sigma_2 = 1 \qquad (119)$$

By subjecting the material to two tension and two compressive and shear tests, coefficients F_i, F_{ij} become

$$\left.\begin{aligned} F_1 &= \frac{1}{X_T} - \frac{1}{X_C}\;; & F_{11} &= \frac{1}{X_T X_C} \\ F_2 &= \frac{1}{Y_T} - \frac{1}{Y_C}\;; & F_{22} &= \frac{1}{Y_T Y_C} \\ F_6 &+ \frac{1}{S^+} - \frac{1}{S^-}\;; & F_{66} &+ \frac{1}{S^+ S^-} \end{aligned}\right\} \qquad (120)$$

Similar expressions can be obtained for F_3, F_4, F_5, etc in case of three dimensional orthotropic material.

To determine F_{12} etc, a variety of combinations of σ_1, and σ_2 can be used. Malmeister also suggests the use of other criteria based on

(a) Ultimate strains; and

(b) Both ultimate stresses and strains.

They are

$$E_{\alpha\beta}\ \varepsilon_{\alpha\beta} + E_{\alpha\beta\gamma\delta}\ \varepsilon_{\alpha\beta}\ \varepsilon_{\gamma\delta} + \ldots\ldots = 1 \text{ and} \tag{121}$$

$$F_{\alpha\beta}\ \sigma_{\alpha\beta} + E_{\alpha\beta}\ \varepsilon_{\alpha\beta} + F_{\alpha\beta\gamma\delta}\ \sigma_{\alpha\beta}\ \sigma_{\gamma\delta} + E_{\alpha\beta\gamma\delta}\ \varepsilon_{\alpha\beta}\ \varepsilon_{\gamma\delta} + \ldots = 1 \tag{122}$$

(10) <u>Hoffman (1967)</u>

To determine the brittle strength of orthotropic material, Hoffman (Reference 33) proposed that a fracture condition is reached when Equation 123 is satisfied, i.e.,

$$C_1(\sigma_{22} - \sigma_{33})^2 + C_2(\sigma_{33} - \sigma_{11})^2 + C_3(\sigma_{11} - \sigma_{22})^2 + C_4\ \sigma_{11}$$

$$+ C_5\ \sigma_{22} + C_6\ \sigma_{33} + C_7\ \tau_{23}^2 + C_8\ \tau_{13}^2 + C_9\ \tau_{12}^2 = 1 \tag{123}$$

The constants C_i's can be expressed in terms of three tensile strengths, three compressive strengths and three shear strengths, i.e.,

$$\left.\begin{aligned}
C_1 &= 1/2\left[\frac{1}{Y_T Y_C} + \frac{1}{Z_T Z_C} - \frac{1}{X_T X_C}\right] \\
C_2 &= 1/2\left[\frac{1}{Z_T Z_C} + \frac{1}{X_T X_C} - \frac{1}{Y_T Y_C}\right] \\
C_3 &= 1/2\left[\frac{1}{X_T X_C} + \frac{1}{Y_T Y_C} - \frac{1}{Z_T Z_C}\right]
\end{aligned}\right| \tag{124}$$

$$C_4 = \frac{1}{X_T} - \frac{1}{X_C}$$

$$C_5 = \frac{1}{Y_T} - \frac{1}{Y_C}$$

$$C_6 = \frac{1}{Z_T} - \frac{1}{Z_C}$$

$$C_7 = \frac{1}{T^2}$$

$$C_8 = \frac{1}{R^2}$$

$$C_9 = \frac{1}{S^2} \qquad (124)$$

In case of plane stress condition and assuming after Reference 25, that $Y_C = Z_C, Y_T = Z_T, T = R$, Equation 123 becomes

$$\frac{\sigma_{11}^2}{X_T X_C} + \frac{\sigma_{22}^2}{Y_T Y_C} - \frac{\sigma_{11}\,\sigma_{22}}{X_T X_C} + \frac{X_C - X_T}{X_T X_C}\sigma_{11} + \frac{Y_C - Y_T}{Y_T Y_C}\sigma_{22} + \frac{\tau_{12}^2}{S^2} = 1 \qquad (125)$$

(11) Fisher (1967)

Using the strength criterion of Reference 23, Fisher (Reference 34) derived expressions to evaluate the strength of isotropic laminates with N laminas($N > 3$) and inclined at an angle of $\frac{\pi}{N}$ angle to each other. The expressior is:

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 - K\,\frac{\sigma_{11}\,\sigma_{22}}{XY} = 1 \qquad (126)$$

where

$$K = \frac{E_{11}(1 + \mu_{21}) + E_{22}(1 + \mu_{12})}{2\sqrt{E_{11} E_{22} (1 + \mu_{12})(1 + \mu_{21})}} \tag{127}$$

(12) Chamis (1967)

Chamis (Reference 35) assumed that

(i) Each ply is generally orthotropic and is linearly elastic;

(ii) Stress strain curves are the same under tensile and compressive loads;

(iii) The ply is conservative under loads ie, the tensor of elastic constants is symmetric;

(iv) The ply experiences only extensional deformation under free thermal loading; and

(v) The distortional energy of each ply remains invariant under rotational transformation.

Following Reference 24, the distortional energy, independent of thermal effects, can be expressed as $U_D = K_1 \sigma_1^2 + K_2 \sigma_2^2 + K_3 \sigma_3^2 + K_4 \sigma_1 \sigma_2$

$$+ K_5 \sigma_2 \sigma_3 + K_6 \sigma_1 \sigma_3 + K_7 \sigma_4^2 + K_8 \sigma_5^2 + K_9 \sigma_6^2 \tag{128}$$

For simple load conditions,

$$U_D = K_1 X^2 = K_2 Y^2 = K_3 Z^2 = K_7 T^2 = K_8 R^2 = K_9 S^2 \tag{129}$$

Using Equation 129 in Equation 128 yields

$$\left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_2}{Y}\right)^2 + \left(\frac{\sigma_3}{Z}\right)^2 + \left(\frac{\sigma_4}{T}\right)^2 + \left(\frac{\sigma_5}{R}\right)^2 + \left(\frac{\sigma_6}{S}\right)^2$$

$$- K_{12} \frac{\sigma_1\sigma_2}{XY} - K_{23} \frac{\sigma_2\sigma_3}{YZ} - K_{13} \frac{\sigma_1\sigma_3}{XZ} = 1 \qquad (130)$$

where K_{12}, K_{23}, and K_{13} are the combined strength coefficients which are chosen in such a manner that predicted and experimental results are in good agreement. For a plane stress condition, Equation 130 assumes the form

$$\left(\frac{\sigma_{11}}{X}\right)^2 + \left(\frac{\sigma_{22}}{Y}\right)^2 + \left(\frac{\tau_{12}}{S}\right)^2 - K_{12} \frac{\sigma_{11}\sigma_{22}}{XY} = 1 \qquad (131)$$

Equation 131 does not distinguish between the tensile and compressive behavior. A distinction can easily be allowed for by permitting X and Y to assume values X_T or X_C Y_T or Y_C consistent with the stresses σ_1 and σ_2.

(13) Bogue (1967)

In the formulation of a yield criterion for orthotropic materials the assumptions made in Reference 36 are:

(i) The yield criterion depends upon the present stress state only, i.e., the stress or strain history does not affect the yield strength;

(ii) The yield criterion is a scalar quantity, i.e., some scalar combination of stresses determines when yield would occur;

(iii) Only the devatoric stresses affect the yield;

(iv) The material has three planes of anisotropy.

The deviatoric stress components are

$$\tau^i_j = \sigma^i_j - \frac{\sigma^k_K}{3} \delta_{ij} \ (i,j,k = 1,2,3) \qquad (132)$$

Truncating the general tensor polynomial after the first cubic term and specialising to orthogonal materials (not necessarily cartesian), the yield condition is written as

$$\alpha_1 \tau_1^1 + \alpha_2 \tau_2^2 + \alpha_3 \tau_3^3 - 1/2 \Big[\gamma_{12}(\tau_1^1 - \tau_2^2)^2 + \gamma_{13}(\tau_1^1 - \tau_3^3)^2$$
$$+ \gamma_{23}(\tau_2^2 - \tau_3^3)^2 \Big] + \beta_{12} \tau_2^1 \tau_1^2 + \beta_{13} \tau_3^1 \tau_1^3 + \beta_{23} \tau_3^2 \tau_2^3 + C$$
$$\det \left[\tau_n^u \right] = 1 \qquad (133)$$

where $(\alpha_1 + \alpha_2 + \alpha_3) = 0$ (134)

Equation 133 has ten constants which can be evaluated from three tension, three compression, three shear tests and using the Equation 134. If $\alpha_1 = \alpha_2 = \alpha_3 = C = 0$, Equation 133 reduces to Hill's six constant theory.

(14) Franklin (1969)

Marin's criterion assumed that the material axes coincided with the principal stress direction. Franklin (Reference 37) generalized it to include shearing stresses on material planes. The expression suggested by him for plane stress condition is

$$K_1 \sigma_{11}^2 + K_2 \sigma_{11} \sigma_{22} + K_3 \sigma_{22}^2 + K_4 \sigma_{11} + K_5 \sigma_{22} + K_6 \tau_{12}^2 = 1 \qquad (135)$$

Constants K's are determined from two tensile, two compressive and one shear test. Using the experimental values strengths obtained under simple loading conditions, Equation 135 becomes,

$$\frac{\sigma_{11}^2}{X_T X_C} + \frac{\sigma_{22}^2}{Y_T Y_C} - \frac{K_2\, \sigma_{11}\, \sigma_{22}}{X_T X_C} + \frac{X_C - X_T}{X_T X_C}\sigma_{11} + \frac{Y_C - Y_T}{Y_T Y_C}\sigma_{22} + \left(\frac{\tau_{12}}{S}\right)^2 = 1 \tag{136}$$

If $K_2 = 1$, Equation 136 reduces to that by Hoffman (Equation 125).

The constant K_2 is a floating constant. Its value may be different in different quandrant of stress space. It can be evaluated from biaxial stress states.

(15) Tsai and Wu (1971)

Assuming that there exists a failure surface in the stress space, Tsai and Wu (Reference 38) propose a scalar function

$$f(\sigma_K) = F_i \sigma_i + F_{ij}\, \sigma_i\, \sigma_j = 1 \quad (i,j = 1,2\ldots.6) \tag{137}$$

subject to the constraint

$$F_{ii}\, F_{jj} - F_{ij}{}^2 \geq 0 \qquad (i,j = 1,2\ldots6) \tag{138}$$

In Equation 138 summations over i and j are not implied.

F_i and F_{ij} are related to the strengths of material obtained from tests conducted with reference to the coordinate system. Three tension and three compressive tests yield X_T, Y_T, Z_T, and X_C, Y_C, Z_C. The use of the results of the tests in Equation 137, yields

$$\begin{aligned} F_1 &= \frac{1}{X_T} - \frac{1}{X_C} \;; \quad F_{11} = \frac{1}{X_T X_C} \\ F_2 &= \frac{1}{Y_T} - \frac{1}{Y_C} \;; \quad F_{22} = \frac{1}{Y_T Y_C} \\ F_3 &= \frac{1}{Z_T} - \frac{1}{Z_C} \;; \quad F_{33} = \frac{1}{Z_T Z_C} \end{aligned} \tag{139}$$

Similarly six pure shear tests (three positive, and three negative shear) are used to obtain

$$F_4 + \frac{1}{T^+} - \frac{1}{T^-} \qquad F_{44} = \frac{1}{T^+T^-}$$

$$F_5 = \frac{1}{R^+} - \frac{1}{R^-} \qquad F_{55} = \frac{1}{R^+R^-} \tag{140}$$

$$F_6 = \frac{1}{S^+} - \frac{1}{S^-} \qquad F_{66} = \frac{1}{S^+S^-}$$

To evaluate the rest of F_{ij}, combined stress tests are required, for example, F_{12}. To determine F_{12}, variety of stress combinations can be used. Some of them are

(i) $\sigma_1 = \sigma_2 = P_T$;

(ii) 45-degree specimen in tension

$$\sigma_1 = \sigma_2 = \sigma_6 = \frac{U_T}{2};$$

(iii) 45-degree specimen in compression

$$\sigma_1 = \sigma_2 = \sigma_6 = \frac{U_C}{2};$$

(iv) $\sigma_1 = -\sigma_2 = V_T$;

(v) $\sigma_1 = -\sigma_2 = -V_C$;

(vi) $\sigma_1 = \sigma_2 = -P_C$;

Plots of P_T, P_C etc. as a function F_{12} indicates that not all the tests are suitable for determining F_{12}. A small inaccuracy in the value of U_T, P_T, V_C produces a large change in the value of F_{12}. For orthotropic materials, results get highly simplified. The number of independent constants which are non-zero, are 12. This reduces Equation 137 to

$$F_1 \sigma_1 + F_2 \sigma_2 + F_3 \sigma_3 + F_{11} \sigma_1^2 + 2F_{12} \sigma_1 \sigma_2 + 2F_{13} \sigma_1 \sigma_3 +$$
$$F_{22} \sigma_2^2 + 2F_{23} \sigma_2 \sigma_3 + F_{33} \sigma_3^2 + \qquad (141)$$
$$F_{44} \sigma_4^2 + F_{55} \sigma_5^2 + F_{66} \sigma_6^2 = 1$$

In case of the generalized plane stress, Equation 141 becomes

$$F_1 \sigma_1 + F_2 \sigma_2 + F_{11} \sigma_1^2 + 2F_{12} \sigma_1 \sigma_2 + F_{22} \sigma_2^2 + F_{66} \sigma_6^2 = 1 \qquad (142)$$

(16) Puppo and Evensen (1971)

It is observed in References 39, 40 that the behavior of orthotropic materials appears to lie between two cases marked by the responses of the isotropic ductile materials and non-interacting fabric like materials. To bring about the transition between the two extreme cases, the concept of interacting factors is introduced. Defining the factors as

$$\begin{aligned} \alpha_1 &= \frac{3T^2}{YZ} \\ \beta_1 &= \frac{3R^2}{XZ} \\ \gamma_1 &= \frac{3S^2}{XY} \end{aligned} \qquad (143)$$

a failure criterion

$$\sigma^T R_1^{(i)} \sigma = 1 \qquad (i = 1,2,3) \qquad (144)$$

is postulated, where σ and $R_1^{(i)}$ are

$$\sigma^T = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6) \qquad (145)$$

$$R_1^{(i)} = \left[\begin{array}{c:c} A_1^{(i)} & 0 \\ \hdashline 0 & B_1 \end{array}\right] \qquad (i = 1,2,3) \qquad (146)$$

$$A_1^{(1)} = \begin{bmatrix} \frac{1}{X^2} & -\frac{\gamma_1}{2Y^2} & -\frac{\beta_1}{2Z^2} \\ & \frac{\gamma_1}{Y^2} & -\frac{1}{2X^2} \\ \text{Sym} & & \frac{\beta_1}{Z^2} \end{bmatrix} \tag{147}$$

$$A_1^{(2)} = \begin{bmatrix} \frac{\gamma_1}{X^2} & -\frac{\gamma_1}{2X^2} & -\frac{1}{2Y^2} \\ & \frac{1}{Y^2} & -\frac{\alpha_1}{2Z^2} \\ \text{Sym} & & \frac{\alpha_1}{Z^2} \end{bmatrix} \tag{148}$$

$$A_1^{(3)} = \begin{bmatrix} \frac{\beta_1}{X^2} & -\frac{1}{2Z^2} & -\frac{\beta_1}{2X^2} \\ & \frac{\alpha_1}{Y^2} & -\frac{\alpha_1}{2Y^2} \\ \text{Sym} & & \frac{1}{Z^2} \end{bmatrix} \tag{149}$$

$$B_1 = \begin{bmatrix} \frac{1}{T^2} & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & \frac{1}{S^2} \end{bmatrix} \tag{150}$$

In case of plane stress, Equation 144 becomes

$$(\frac{\sigma_{11}}{X})^2 - \gamma_1 (\frac{X}{Y} \frac{\sigma_{11}}{X} \frac{\sigma_{22}}{Y}) + \gamma_1 (\frac{\sigma_{22}}{Y})^2 + \frac{\tau_{12}^2}{S^2} = 1 \qquad (151)$$

$$\gamma_1 (\frac{\sigma_{11}}{X})^2 - \gamma_1 (\frac{Y}{X} \frac{\sigma_{11}}{X} \frac{\sigma_{22}}{Y}) + (\frac{\sigma_{22}}{Y})^2 + \frac{\tau_{12}^2}{S^2} = 1 \qquad (152)$$

For $X = Y = \sigma_0$ and $S = \frac{\sigma_0}{3}$, $\gamma_1 = 1$ and Equations 151 and 152 yield identical results.

Figure 9 indicates the effect of interacting factor γ_1 on the shape of failure surface for $\tau_{12} = 0$.

SECTION IV

SUMMARY

In Sections II and III various failure theories for isotropic and anisotropic materials have been surveyed. The criteria discussed in Section III are either distinct failure-mode-dependent or have gradual failure mode transitions. In the first category, failure is precipitated when any one or all of longitudinal, transverse, and shear stresses/strains (References 10, 15) exceed the limits determined by tests. Tests (References 12,13) conducted on off-axis coupons of a thin composite material, however, did not indicate any peak when the mode of failure changed from shear to transverse. It appears to indicate that shear and transverse failure modes of composites with stiff fibers are not independent but interact (Reference 14). If the fibers are stiff but ductile, the failure condition of the composite, according to Reference 18, may be predicted by considering three shear modes of failure.

The second class of criteria are essentially different expressions of a quadratic form with or without linear terms. They are either generalizations of Von Mises' criterion (References 19, 20, 21, 22, 23, 24, 25, 33, 34, 35, 37) or have been developed explicitly in quadratic form using the stress tensor (Reference 26, 32, 38, 39) or the stress deviation (Reference 36). In the latter case one cubic term is also included.

If differences in tensile and compressive strengths are not explicitly allowed for, the expression for the failure criterion can be written as

$$F_{ij}\,\sigma_i\,\sigma_j = 1 \qquad (i,j = 1,2\ldots6) \tag{153}$$

or

$$\sigma^T F\,\sigma = 1 \tag{154}$$

where σ^T is the transpose of σ.

In case differences in strength are expressedly accounted for, the criterion is

$$F_j\,\sigma_j + F_{ij}\,\sigma_i\,\sigma_j = 1 \qquad (i,j = 1,2\ldots6) \tag{155}$$

or

$$(\bar{F}^T + \sigma^T F)\,\sigma = 1 \tag{156}$$

If $\bar{F}$ and F are known in one coordinate system, they can be determined in any other system by suitable transformation of coordinates. The coefficients $\bar{F}$, F strength parameters of various theories for the orthotropic sheet in a state of generalized plane stress are summarized in Table I.

REFERENCES

1. Timoshenko, S. P., "History of Strength of Materials", Mcgraw-Hill Book Company, 1953.

2. Nadai, A., "Theory of Flow and Fracture of Solids," Vol I and II, Mcgraw-Hill Book Company.

3. Liebowitz, H., "Frachure," Vol II, Academic Press, 1968.

4. McClintock, F. A., Argon, A. S., "Mechanical Behavior of Materials," Addison-Wesley Publishing Company, Inc., 1966.

5. Jaeger, J. C., "Elasticity, Fracture and Flow," Methuen and Co. Ltd, London, 1962.

6. Bert, C. W., "Biaxial Properties of Metals for Aerospace Structures," AIAA Launch and Space Vehicle Shell Structures Conference, Palm Springs, California, April 1-3, 1963.

7. Timoshenko, S. P., "Strength of Materials," Part II, D. Van Nostrand Company, Inc., 1956.

8. Jenkins, C. F., "Report on Materials of Construction Used in Aircraft and Aircraft Engines," Great Brittain Aeronautical Research Committee, 1920.

9. Kaminski, B. E., and Lantz, R. B., " Composite Materials: Testing and Design, ASTM STP 460, ASTM, 1969.

10. Stowell, E. Z., and Liu, T. S., "On the Mechanical Behavior of Fiber-Reinforced Crystalline Materials," J. Mech. Phys. Solids, Vol 9, 1961

11. Kelly, A. and Davies, G. J., Metall. Rev. 10, 1, 1965.

12. Jackson, P. W., and Cratchley, D.,"The Effect of Fiber Orientation on the Tensile Strength of Fibre-Reinforced Metals," J. Mech. Phys. Solids, Vol 14, 1966.

13. Cooper, G. A., "Orientation Effects in Fibre-Reinforced Metals," J. Mech. Phys. Solids, Vol 14, 1966.

14. Prager, W., "Plastic Failure of Fiber-reinforced Materials," Trans. American Society of Mechanical Engineers, E 36, September 1969.

15. Waddoups, M. E., "Advanced Composite Material Mechanics for the Design and Stress Analyst," General Dynamics, Ft Worth Division, Report FZM-47[illegible], 1967.

16. Hu, L. W., "Modified Tresca's Yield Condition and Associated Flow Rules for Anisotropic Materials and Application," Journal of the Franklin Institute, 1958.

17. Wasti, S. T., "The Plastic Bending of Transversely Anisotropic Circular Plates," Inst. J. Mech. Science, Vol 12, 1970.

18. Lance, R. H., Robinson, D. N., "A Maximum Shear Stress Theory of Plastic Failure of Fiber-reinforced Materials," J. Mech. Phys. Solids, Vol 19, 1971.

19. Hill, R., "A Theory of the Yielding and Plastic Flow of Anisotropic Metals," Proceedings of the Royal Society, Series A, Vol 193, 1948.

20. Hill, R., "The Mathematical Theory of Plasticity," Oxford University Press, London, 1950.

21. Marin J., "Theories of Strength for Combined Stresses and Non-Isotropic Materials," Journal Aeronautical Sciences, 24 (4), April 1957.

22. Stassi-D'Alia, F., "Limiting Conditions of Yielding of Thick Walled Cylinders and Spherical Shells,:, U.S. Army European Research Office EVC-1351 01-4263-60, 24 November, 1959.

23. Norris, C. B., "Strength of Orthotropic Materials Subjected to Combined Stress," Forest Products Laboratory, Report 1816, 1962.

24. Griffith, J. E., and Baldwin, W. M., "Failure Theories for Generally Orthotropic Materials," Developments in Theoretical and Applied Mechanics, Vol 1, 1962.

25. Azzi, V. D., Tsai, S. W., "Anisotropic Strength of Composites," Experimental Mechanics, September 1965.

26. Goldenblat, I. I., and V. A. Kopnov," Strength of Glass-Reinforced Plastics in Complex Stress State," Mekhanika Polimerov, Vol 1 (1965) page 70; English Translation: Polymer Mechanics, Vol 1, 1966.

27. Protasov, V. D., and Kopnov, V. A., "Study of the Strength of Glass-Reinforced Plastics in the Plane Stress State," Mekhanika Polimerov, Vol. 1, No. 5, pp 39-44, 1965.

28. Ashkenazi, E. K., "On the Problem of Strength Anisotropy of Construction Materials", The S. M. Kirov Forest Products Academy, Chair of Construction Mechanics, Leningrad.

29. Ashkenazi, E. K., "Anisotropy in the Strength of Construction Material," C. M. Kirov Wood Technology Academy, Leningrad, Vol 31, No. 5, May 1961.

30. Ashkenazi, E. K., "The Construction of Limiting Surfaces for Biaxial Stressed Condition of Anisotropic Materials," Zavodskaya Laboratoriya, Vol 30, No. 2, Leningrad Forestry Academy, February 1964.

31. Ashkenazi, E. K., "Problems of the Anisotropy of Strength," Mekhanika Polimerov, Vol 1, No. 2, 1965.

32. Malmeister, A. K., "Geometry of Theories of Strength," Mekhanika Polimerov, Vol 2, No. 4, 1966.

33. Hoffman, O., "The Brittle Strength of Orthotropic Materials," J. Composite Materials, Vol 1, 1967.

34. Fischer, L., "Optimization of Orthotropic Laminates," Journal of Engineering for Industry, August 1967.

35. Chamis, C. C., "Failure Criteria for Filamentary Composites," Composite Materials: Testing and Design, ASTM STP 460, ASTM, 1969.

36. Bogue, D. C., "The Yield Stress and Plastic Strain Theory for Anisotropic Materials," ORNL-TM-1869, Oak Ridge National Laboratory, Oak Ridge, Tennessee, July, 1967.

37. Franklin, H. G., "Classic Theories of Failure of Anisotropic Materials," Fiber Science Technology, Vol 1, 1968.

38. Tsai, S. W.., and Wu, E. M., "A General Theory of Strength for Anisotropic Materials," Journal of Composite Materials, Vol 5, January 1971.

39. Puppo, A. H., Evensen, H. A., "Strength of Anisotropic Materials Under Combined Stresses," AIAA/ASME 12th Structures, Structural Dynamics and Materials Conference, Anaheim, California, April 19-21 1971.

40. Puppo, A. H., "General Theory of Failure for Orthotropic Materials," Engineering Report 69-E1, Whittaker Corp., April 1969.

41. Tsai, S. W., "Strength Characteristics of Composite Materials," NASA CR-224, April 1965.

42. Sendeckyj, G. P., "A Brief Survey of Empirical Multiaxial Strength Criteria for Composites,": Testing and Design (Second Conference), ASTM STP 497, pages 41-51, April 1971.

43. Sendeckyj, G. P., "On Empirical Strength Theories," Presented at ASTM-NMAB Symposium on Predictive Testing, Atlantic City, N. J., June-July, 1971.

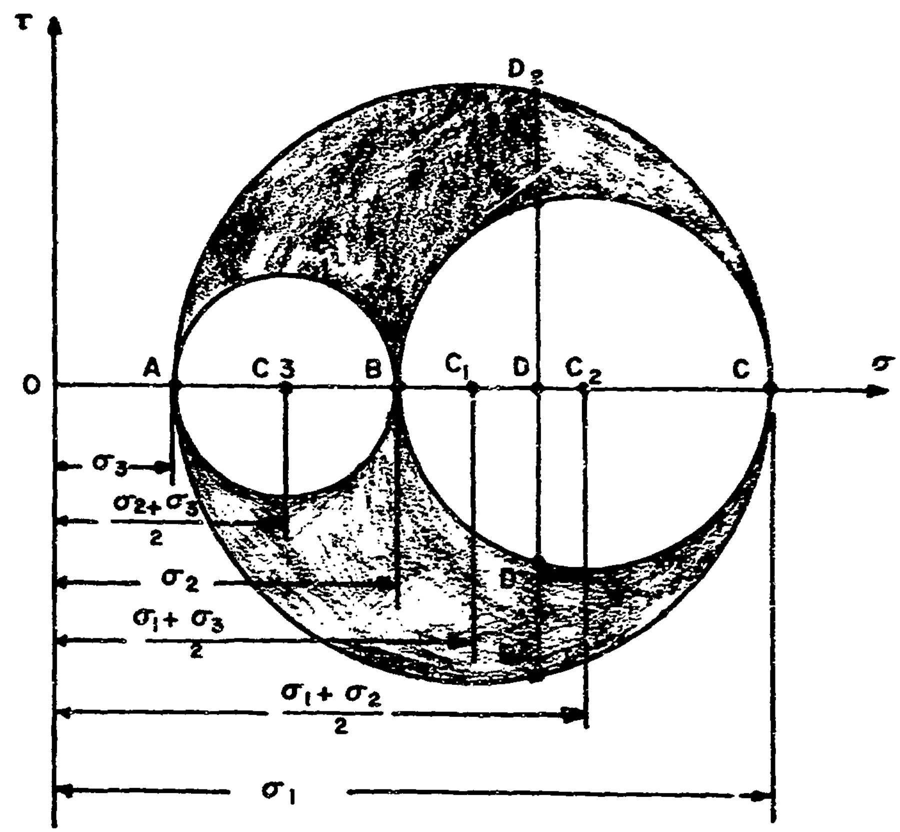

Figure 1. Mohr's Representation of 3-Dimensional Stress State

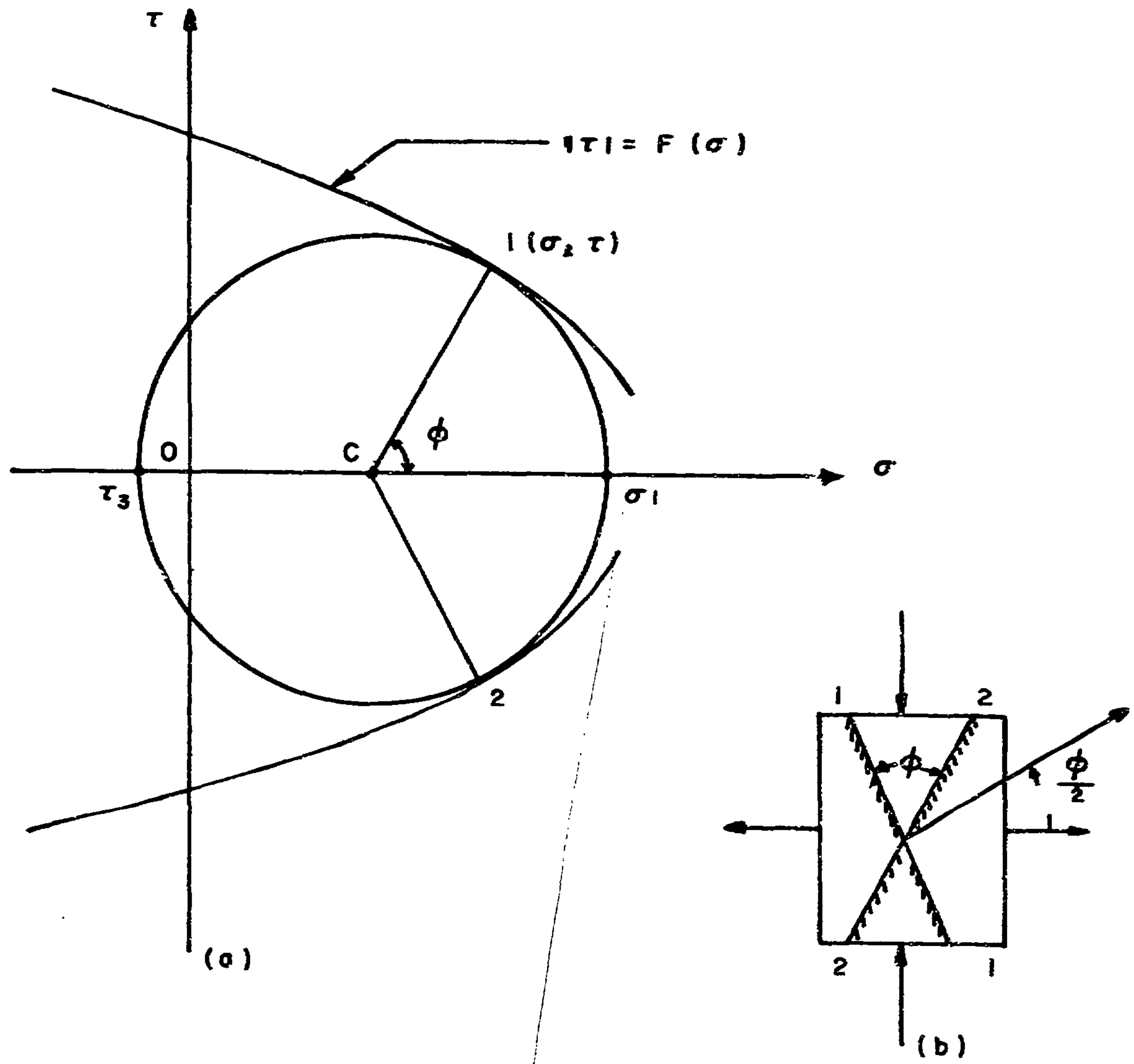

Figure 2. Mohr's Envelope

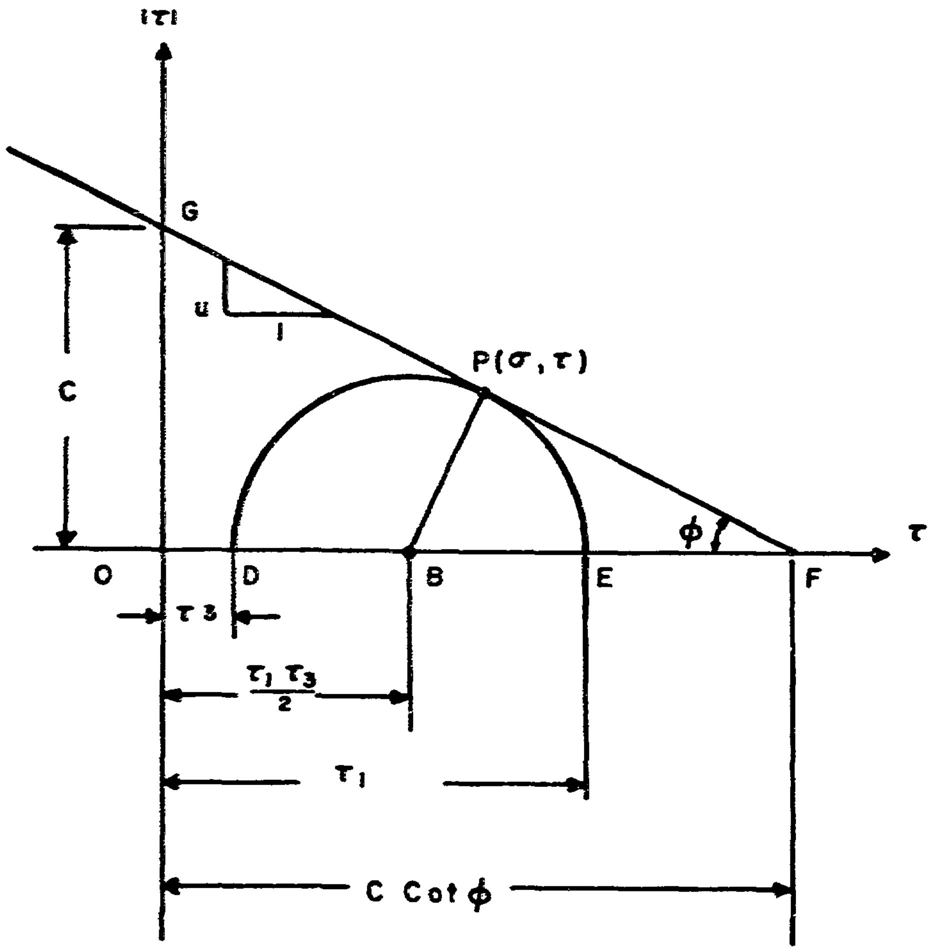

Figure 3. Simplified Mohr's Envelope

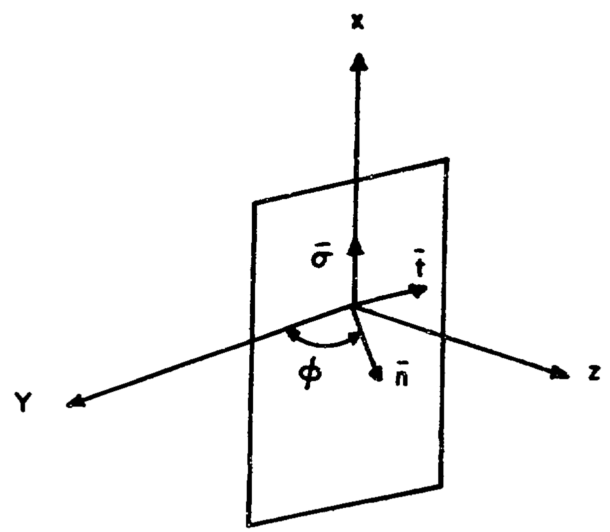

Figure 4. Coordinate Axes

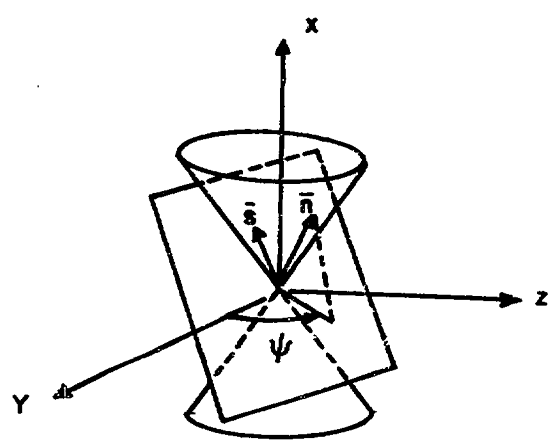

Figure 5. Failure Cone

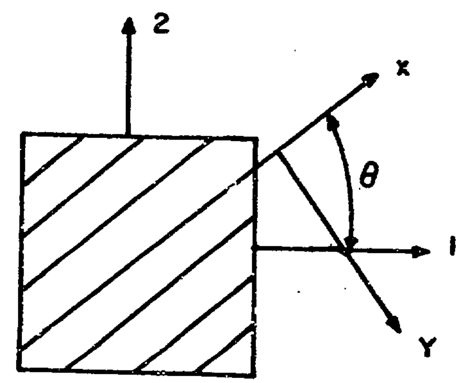

Figure 6. Fiber Orientation

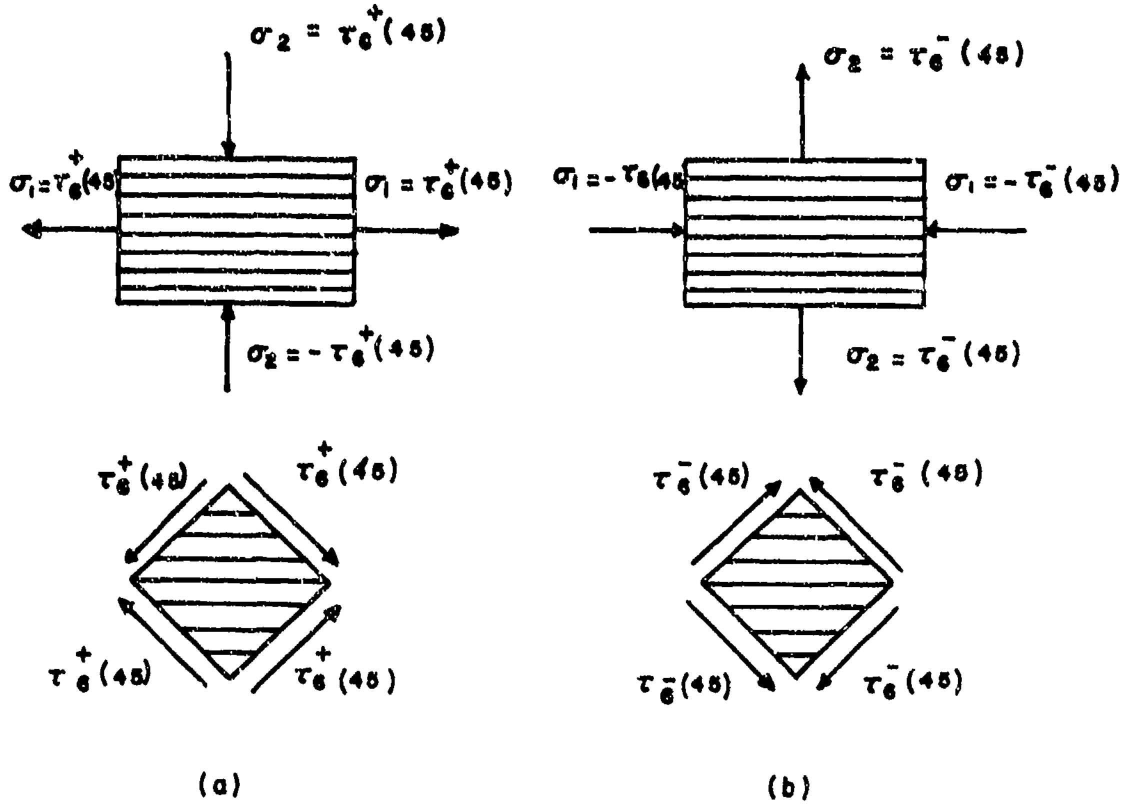

Figure 7. Sign Dependence of Shear Strength

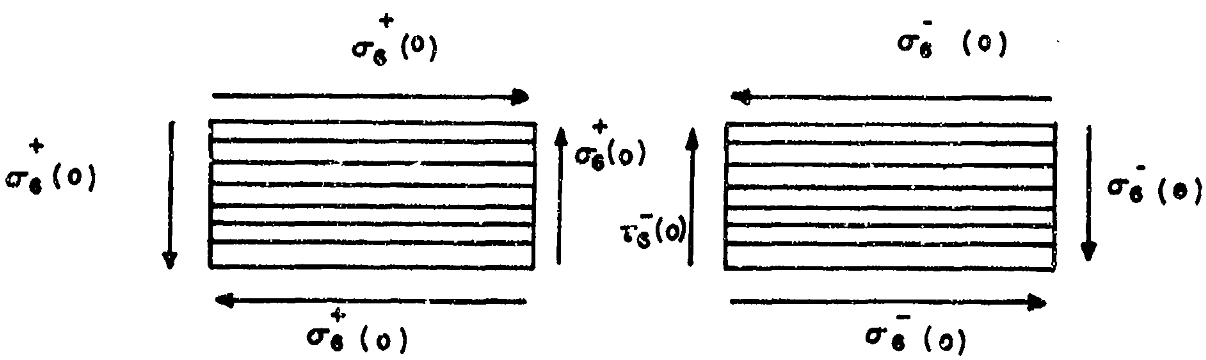

Figure 8. Shear Stresses in Fundamental Coordinates

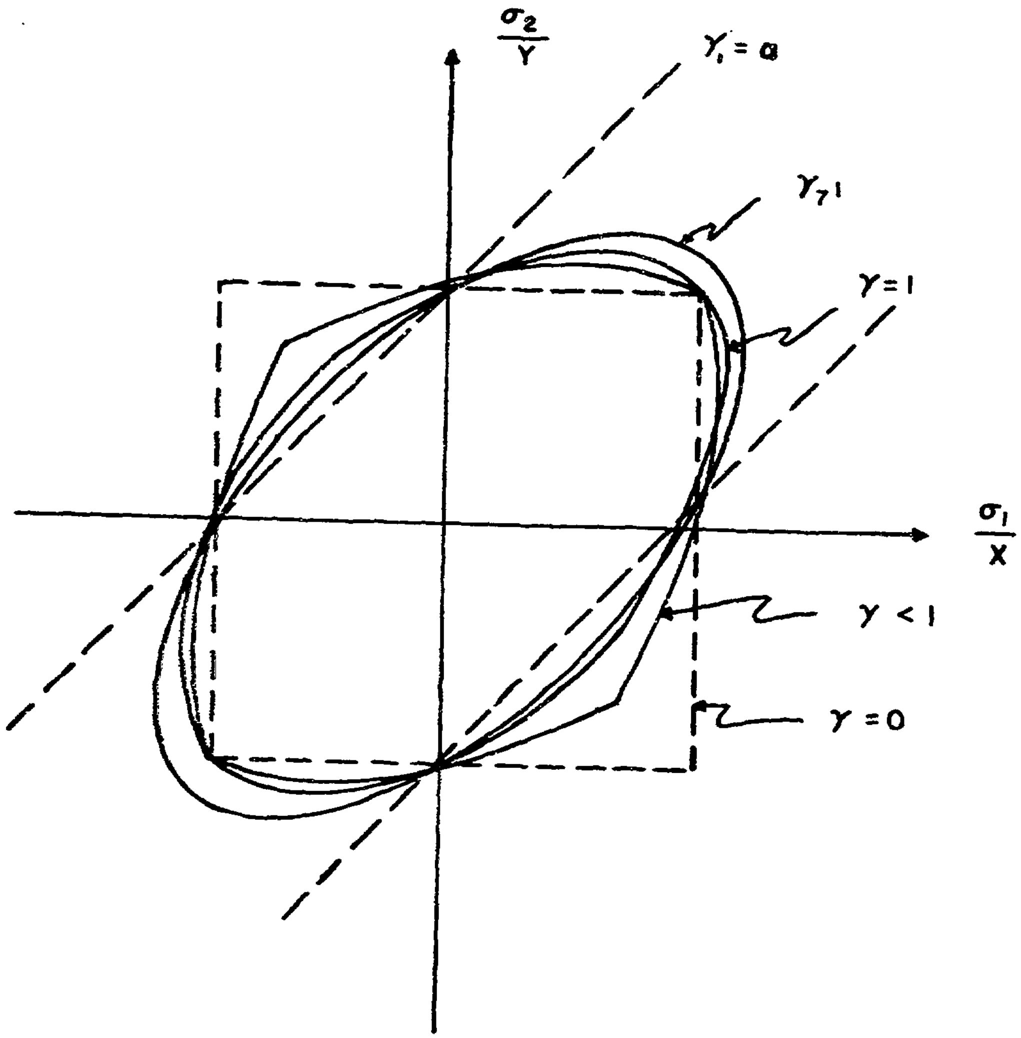

Figure 9. Influence of γ_1 on the Shape of Failure Surface ($\tau_{12} = 0$)

TABLE I

STRENGTH PARAMETERS OF THEORIES
WITHOUT DISTINCT FAILURE MODES FOR PLANE STRESS CONDITION

AUTHORS	EQUATION	F_1	F_2	F_{11}	F_{12}	F_{22}	F_{66}	C	REMARKS
HILL	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{1}{2}\left[\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right]$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		
MARIN	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{1}{X_c X_T}\left[\frac{X_c X_T}{Y_T} - Y_T\right]$	$\frac{1}{X_c X_T}$	$-\frac{K_1}{2X_T X_c}$	$\frac{1}{X_T X_c}$			$K_1 = 2 - \frac{1}{S^2}\left[X_T X_c - S\left(X_c - X_T - X_c\frac{X_T}{Y_T} + Y_T\right)\right]$
STASSI	$F^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{1}{X_c X_T}$	$-\frac{1}{2X_T X_c}$	$\frac{1}{X_T X_c}$	$\frac{3}{X_T^2}$		
NORRIS	$\sigma_1^2/X^2 = 1$, $\sigma_2^2/Y^2 = 1$, $\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{1}{2XY}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		
GRIFFITH - BALDWIN	THIS CRITERION IS ONE MATERIAL CONSTANT CRITERION. ENERGY OF DISTORTION UNDER COMPLEX STRESS IS EQUATED TO THE DISTORTIONAL ENERGY IN A TENSION TEST.								
AZZI- TSAI	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{1}{2X^2}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		
GOLDENBLAT-KOPNOV	$F^T\sigma + \sqrt{\sigma^T F \sigma} = 1$	$\frac{X_c - X_T}{2X_c X_T}$	$\frac{Y_c - Y_T}{2Y_c Y_T}$	$\frac{1}{4}\left[\frac{1}{X_T}+\frac{1}{X_c}\right]^2$	$\frac{1}{8}\left[\left(\frac{1}{X_T}-\frac{1}{X_c}\right)^2+\left(\frac{1}{Y_T}-\frac{1}{Y_c}\right)^2 - \left(\frac{1}{\tau(45)}+\frac{1}{\tau(-45)}\right)^2\right]$	$\frac{1}{4}\left[\frac{1}{Y_T}+\frac{1}{Y_c}\right]^2$	$\frac{1}{4S^2}$		
ASHKENAZI	BIQUADRATIC FORM IS SUGGESTED. THREE CONSTANTS ARE REQUIRED TO BE DETERMINED FROM BIAXIAL TESTS.								
MALMEISTER	$F^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	TO BE DETERMINED FROM BIAXIAL TESTS	$\frac{1}{Y_T Y_c}$	$\frac{1}{S^2}$		
HOFFMAN	$F^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	$-\frac{1}{2X_T X_c}$	$\frac{1}{Y_T Y_c}$	$\frac{1}{S^2}$		
FISCHER	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{K}{2XY}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		$K = \frac{E_1(1+\mu_{21}) + E_{2}(1+\mu_{12})}{2\sqrt{E_1 E_2 (1+\mu_{21})(1+\mu_{12})}}$
CHAMIS	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{K_{12}}{2XY}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		K_{12} - FLOATING CONSTANT
BOGUE	THIS CRITERION HAS A CONSTANT C ASSOCIATED WITH $DET(\tau_m^n)$. FOR THREE DIMENSIONAL ORTHOTROPIC MATERIAL, THERE ARE NINE CONSTANTS TO BE DETERMINED.								
FRANKLIN	$F^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	$-\frac{K_1}{2X_T X_c}$	$\frac{1}{Y_T Y_c}$	$\frac{1}{S^2}$		K_1 - FLOATING CONSTANT
TSAI- WU	$F^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	TO BE DETERMINED FROM BIAXIAL TESTS	$\frac{1}{Y_T Y_c}$	$\frac{1}{S^2}$		
PUPPO- EVENSEN	$\sigma^T F \sigma = 1$ ω ω_1			$\frac{1}{X^2}$ γ_1/X^2	$-\gamma_1/2Y^2$ $-\gamma_1/2X^2$	γ_1/Y^2 $1/Y^2$	$1/S^2$ $1/S^2$		$\gamma_1 = \frac{3S^2}{XY}$

See the following pages for greater detail.

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TABLE I

STRENGTH PARAMETERS OF THE[...]
WITHOUT DISTINCT FAILURE MODES FOR PL[...]

AUTHORS	EQUATION	F_1	F_2	F_{11}	F_{12}
HILL	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{1}{2}\left[\frac{1}{X^2}+\frac{1}{Y^2}\right.$
MARIN	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{1}{X_c X_T}\left[\frac{X_c X_T}{Y_T} - Y_T\right]$	$\frac{1}{X_c X_T}$	$-\frac{K_1}{2 X_T X_c}$
STASSI	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{1}{X_c X_T}$	$-\frac{1}{2 X_T X_c}$
NORRIS	$\sigma_{11}^2/X^2 = 1$ $\sigma_{22}^2/Y^2 = 1$ $\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{1}{2XY}$
GRIFFITH - BALDWIN	THIS CRITERION IS ONE MATERIAL CONSTANT CRITERION. ENERGY OF DISTO[...] UNDER COMPLEX STRESS IS EQUATED TO THE DISTORTIONAL ENERGY IN A TENSI[...]				
AZZI - TSAI	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{1}{2X^2}$
GOLDENBLAT - KOPNOV	$\bar{F}^T\sigma + \sqrt{\sigma^T F \sigma} = 1$	$\frac{X_c - X_T}{2 X_c X_T}$	$\frac{Y_c - Y_T}{2 Y_c Y_T}$	$\frac{1}{4}\left[\frac{1}{X_T}+\frac{1}{X_c}\right]^2$	$\frac{1}{8}\left[\left(\frac{1}{X_T}-\frac{1}{X_c}\right)^2+\right.$ $-\left(\frac{1}{\tau_{(45)}}+\right.$
ASHKENAZI	BIQUADRATIC FORM IS SUGGESTED. THREE CONSTANTS ARE REQUIRED TO BE DETERMINED FROM BIAXIAL TESTS.				
MALMEISTER	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	TO BE DETERMINED [...] BIAXIAL TESTS
HOFFMAN	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	$-\frac{1}{2 X_T X_c}$
FISHER	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{K}{2XY}$
CHAMIS	$\sigma^T F \sigma = 1$			$\frac{1}{X^2}$	$-\frac{K_{12}}{2XY}$
BOGUE	THIS CRITERION HAS A CONSTANT C ASSOCIATED WITH DET(τ_m^n). FOR THREE DIMENSIONAL ORTHOTROPIC MATERIAL, THERE ARE NINE CONSTANTS TO BE DET[...]				
FRANKLIN	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	$-\frac{K_2}{2 X_T X_c}$
TSAI - WU	$\bar{F}^T\sigma + \sigma^T F \sigma = 1$	$\frac{X_c - X_T}{X_c X_T}$	$\frac{Y_c - Y_T}{Y_c Y_T}$	$\frac{1}{X_c X_T}$	TO BE DETERMIN[...] FROM BIAXIAL TE[...]
PUPPO - EVENSEN	$\sigma^T F \sigma = 1$ (i) (ii)			$\frac{1}{X^2}$ γ_1/X^2	$-\gamma_1/2Y^2$ $-\gamma_1/2X^2$

B

TABLE I

...RENGTH PARAMETERS OF THEORIES

...ISTINCT FAILURE MODES FOR PLANE STRESS CONDITION

	F_{11}	F_{12}	F_{22}	F_{66}	C	REMARKS
	$\frac{1}{X^2}$	$-\frac{1}{2}\left[\frac{1}{X^2}+\frac{1}{Y^2}-\frac{1}{Z^2}\right]$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		
... $-Y_T]$	$\frac{1}{X_C X_T}$	$-\frac{K_1}{2X_T X_C}$	$\frac{1}{X_T X_C}$			$K_1 = 2-\frac{1}{S^2}\left[X_T X_C - S(X_C - X_T - X_C\frac{X_T}{Y_T}+Y_T)\right]$
... Y_T	$\frac{1}{X_C X_T}$	$-\frac{1}{2X_T X_C}$	$\frac{1}{X_T X_C}$	$\frac{3}{X_T^2}$		
	$\frac{1}{X^2}$	$-\frac{1}{2XY}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		
...NT CRITERION. ENERGY OF DISTORTION ...E DISTORTIONAL ENERGY IN A TENSION TEST.						
	$\frac{1}{X^2}$	$-\frac{1}{2X^2}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		
... $\frac{Y_T}{Y_T}$	$\frac{1}{4}\left[\frac{1}{X_T}+\frac{1}{X_C}\right]^2$	$\frac{1}{8}\left[\left(\frac{1}{X_T}-\frac{1}{X_C}\right)^2+\left(\frac{1}{Y_T}-\frac{1}{Y_C}\right)^2 - \left(\frac{1}{\tau_6(45^\circ)^+}+\frac{1}{\tau_6(45^\circ)^-}\right)^2\right]$	$\frac{1}{4}\left[\frac{1}{Y_T}+\frac{1}{Y_C}\right]^2$	$\frac{1}{4S^2}$		
...HREE CONSTANTS ...BIAXIAL TESTS.						
... $\frac{Y_T}{Y_T}$	$\frac{1}{X_C X_T}$	TO BE DETERMINED FROM BIAXIAL TESTS	$\frac{1}{Y_T Y_C}$	$\frac{1}{S^2}$		
... $\frac{Y_T}{Y_T}$	$\frac{1}{X_C X_T}$	$-\frac{1}{2X_T X_C}$	$\frac{1}{Y_T Y_C}$	$\frac{1}{S^2}$		
	$\frac{1}{X^2}$	$-\frac{K}{2XY}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		$K = \frac{E_{11}(1+\mu_{21}) + E_{22}(1+\mu_{12})}{2\sqrt{E_{11}E_{22}(1+\mu_{21})(1+\mu_{12})}}$
	$\frac{1}{X^2}$	$-\frac{K_{12}}{2XY}$	$\frac{1}{Y^2}$	$\frac{1}{S^2}$		K_{12} - FLOATING CONSTANT
...IATED WITH $\det(\tau_m^n)$. FOR THREE ...ERE ARE NINE CONSTANTS TO BE DETERMINED.						
... $\frac{Y_T}{Y_T}$	$\frac{1}{X_C X_T}$	$-\frac{K_2}{2X_T X_C}$	$\frac{1}{Y_T Y_C}$	$\frac{1}{S^2}$		K_2 - FLOATING CONSTANT
... $\frac{Y_T}{Y_T}$	$\frac{1}{X_C X_T}$	TO BE DETERMINED FROM BIAXIAL TESTS	$\frac{1}{Y_T Y_C}$	$\frac{1}{S^2}$		
	$\frac{1}{X^2}$ γ_1/X^2	$-\gamma_1/2Y^2$ $-\gamma_1/2X^2$	γ_1/Y^2 $1/Y^2$	$1/S^2$ $1/S^2$		$\gamma_1 = \frac{3S^2}{XY}$